School of Mathematics and Statistics
Faculty of Science, Technology, Engineering and Mathematics

2018 PhD Projects

Project title	Renormalization for area-preserving maps
Principal supervisor	Ben Mestel
Second supervisor	Ian Short
Discipline	Pure Mathematics
Research area/keywords	Dynamical systems, analysis, renormalization, functional equations
Suitable for	Full time applicants

Project background and description

The study (via renormalization) of universal properties of chaotic transitions (such as the celebrated Feigenbaum period-doubling cascade for unimodal maps) leads to nonlinear functional equations that are particularly hard to analyse.

Area-preserving maps, such as the famous standard map or Taylor-Chirikov map on the cylinder $S^1 \times \mathbb{R}$

$$T_k(x, y) = \left(x + y - \frac{k}{2\pi} \sin 2\pi x, \quad y - \frac{k}{2\pi} \sin 2\pi x \right),$$

are important models for the understanding of Hamiltonian systems in general. Here $k \geq 0$ is a parameter. For $k = 0$, the system is integrable (solvable), but for $k > 0$ the system is non-integrable with islands of stability interspersed with chaotic regions.

Invariant curves which wrap round the cylinder are important to understand the global stability of Hamiltonian systems, and can be understood in terms of their rotation numbers, the average number of turns around the cylinder for each iteration. For arithmetically 'good' irrational rotation numbers, invariant curves are known to exist for small k (by the celebrated Kolmogorov-Arnol'd-Moser Theorem) but are progressively destroyed as k increases. For the standard map, the last such invariant curve (corresponding to golden-ratio rotation number) breaks up at a critical value $k = k_c \approx 0.971635$.

The theory (known as renormalization) that describes the properties of the final golden-mean invariant circle has been known since the 1980s and so far only computer-assisted proofs have been available to analyse the solutions of the associated functional equations. This project will use real and complex analytical techniques developed in dynamical systems over the past 20+ years to study the renormalisation functional equations for 'good' irrationals with a view to developing an analytic proof of the existence of a solution of the golden-mean functional equations.

The project requires knowledge of analysis and of dynamical systems theory. You will work closely with your supervisors on a day-to-day basis.

Background reading/references

- R.S MacKay and J.D Meiss, Hamiltonian Dynamical Systems: A reprint selection, CRC Press 1987.

- H. Koch, *A Renormalization Group Fixed Point Associated with the Breakup of Golden Invariant Tori*, Discrete and Continuous Dynamical Systems 11(4) (2004), 881-909.