School of Mathematics and Statistics

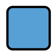
Faculty of Science, Technology, Engineering and Mathematics

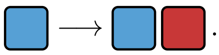
2018 PhD Projects

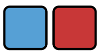
Project title	Substitution dynamics and semigroups
Principal supervisors	Uwe Grimm, Ian Short
Discipline	Pure mathematics
Research area/keywords	dynamical systems, symbolic dynamics, spectral theory
Suitable for	Full-time or part-time applicants

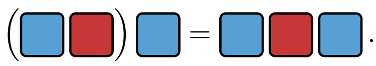
Project background and description

This project is about substitution dynamical systems, one of the simplest of which can be described graphically as follows. Let f be the transformation that satisfies the rules

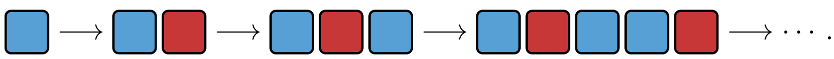
Now consider the pattern that emerges if we apply f repeatedly, starting from a single tile . The first step is



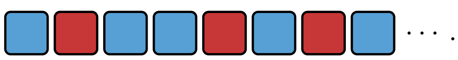
For the next step, we apply f to each tile of  separately, to give



Continuing in this way, and making some natural assumptions about composition, we obtain



Proceeding in this fashion indefinitely, we end up with an infinite sequence of tiles



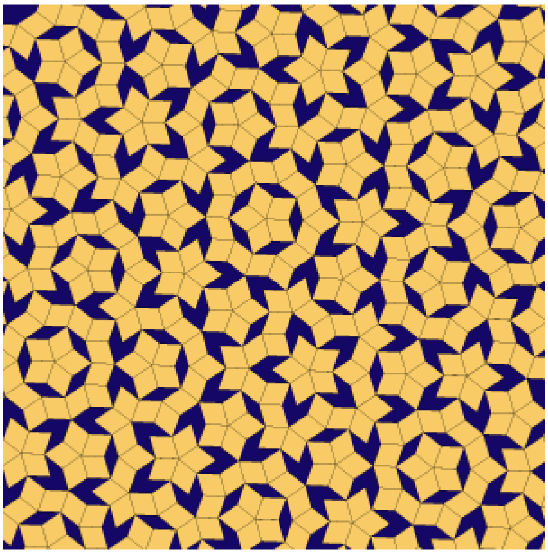
The resulting tiling is said to be *aperiodic* because it has no translational symmetries. The transformation f is called the *Fibonacci substitution* because after applying this transformation n times there are F_n red tiles, F_{n+1} blue tiles, and F_{n+2} tiles in total, where $F_1, F_2, F_3, \ldots$ is the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots,$$

which satisfies the recurrence relation $F_{n+2} = F_{n+1} + F_n$.

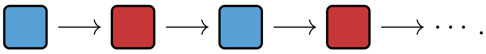
Tilings of this type really come to life in two dimensions, where, by similar methods, we can obtain beautiful aperiodic structures such as the one below.

This aperiodic tiling, taken from the online Tilings Encyclopedia, was first constructed by the mathematical physicist Roger Penrose in the early 1970s. The work of Penrose and others has attracted a great deal of interest in mathematics, partly because of the rich geometric and dynamic structure of these tilings, but also because of their relationship to remarkable phyiscal objects called *quasicrystals*, the discovery of which earned the material scientist Dan Shechtman the Nobel Prize for Chemistry in 2011.

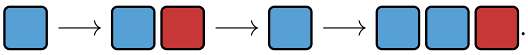


Recent research has examined the effect of combining substitutions. Consider, for example, the simple substitution g that satisfies

Applying g repeatedly, starting from a single tile , gives rise to the alternating sequence



What is more intriguing is the tiling that emerges if we apply a sequence of the transformations f and g in some order. For instance, if we apply f, then $f \circ g$, then $f \circ g \circ f$ to the starting tile then we obtain the pattern



More generally, given a sequence $H_n = h_1 \circ h_2 \circ \cdots \circ h_n$, where h_i is equal to either f or g for $i = 1, 2, \ldots$, we can study the tilings that emerge as images of under (H_n).

This project is about the dynamics, geometry and spectal properties of mixed substitutions of this type. We will investigate how the behaviour of sequences such as (H_n) relates to the semigroup generated by f and g, which comprises all possible compositions of words in f and g. This collection of related objects is rich in arithmetic and geometric structure, and it feeds into a variety of other mathematical disciplines, including dynamical systems, continued fractions theory, and fractal geometry.

Background reading/references

- M. Baake and U. Grimm, *Aperiodic order. Vol. 1*, Encyclopedia of Mathematics and its Applications, 149, Cambridge University Press, Cambridge, 2013.
- M. Baake, D. Damanik and U. Grimm, *What is . . . aperiodic order?*, Notices Amer. Math. Soc. **63** (2016), no. 6, 647–650. https://arxiv.org/pdf/1512.05104.pdf
- M. Baake, D. Damanik and U. Grimm, *Aperiodic order and spectral properties*. https://arxiv.org/pdf/1506.04978.pdf