Permutations and permutation graphs

Robert Brignall

Schloss Dagstuhl, 8 November 2018
Permutations and permutation graphs

• Permutation $\pi = \pi(1) \cdots \pi(n)$

• Inversion graph $G_{\pi}$: for $i < j$, $ij \in E(G_{\pi})$ iff $\pi(i) > \pi(j)$.

• Note: $n \cdots 21$ becomes $K_n$.

• Permutation graph = can be made from a permutation
Permutations and permutation graphs

- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Inversion graph $G_\pi$: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes $K_n$.
- Permutation graph $=$ can be made from a permutation
Ordering permutations: containment

• ‘Classical’ pattern containment: $\sigma \leq \pi$.
• Translates to induced subgraphs: $G_\sigma \leq_{\text{ind}} G_\pi$.
• Permutation class: a downset:
  $$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$  
• Avoidance: minimal forbidden permutation characterisation:
  $$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$
Ordering permutations: containment

- ‘Classical’ pattern containment: $\sigma \leq \pi$.
- Translates to induced subgraphs: $G_{\sigma} \leq_{\text{ind}} G_{\pi}$.
- Permutation class: a downset:
  \[ \pi \in C \text{ and } \sigma \leq \pi \implies \sigma \in C. \]
- Avoidance: minimal forbidden permutation characterisation:
  \[ C = \text{Av}(B) = \{ \pi : \beta \nleq \pi \text{ for all } \beta \in B \}. \]
Ordering permutations: containment

• ‘Classical’ pattern containment: $\sigma \leq \pi$.
• Translates to induced subgraphs: $G_\sigma \leq_{\text{ind}} G_\pi$.
• Permutation class: a downset:

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$  

• Avoidance: minimal forbidden permutation characterisation:

$$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$
• **Induced subgraph:** $H \leq_{\text{ind}} G$: ‘delete vertices’ (& incident edges).

• **Hereditary class:** $\mathcal{C}$, a downset:

$$G \in \mathcal{C} \text{ and } H \leq_{\text{ind}} G \implies H \in \mathcal{C}.$$  

(Example: all planar graphs.)

• **Forbidden induced subgraph characterisation**, $\text{Free}(G_1, \ldots, G_k)$. 
Ordering graphs: induced subgraphs

- **Induced subgraph:** $H \leq_{\text{ind}} G$: ‘delete vertices’ (& incident edges).
- **Hereditary class:** $\mathcal{C}$, a downset:

  $$G \in \mathcal{C} \text{ and } H \leq_{\text{ind}} G \implies H \in \mathcal{C}.$$  

  (Example: all planar graphs.)
- **Forbidden induced subgraph characterisation,** $\text{Free}(G_1, \ldots, G_k)$.
<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutation ( \pi )</td>
<td>Permutation graph ( G_\pi )</td>
</tr>
<tr>
<td>Containment ( \pi &lt; \sigma )</td>
<td>Induced subgraph ( G_\pi &lt;<em>{\text{ind}} G</em>\sigma )</td>
</tr>
<tr>
<td>Class ( \mathcal{C} )</td>
<td>Class ( G_\mathcal{C} = { G_\pi : \pi \in \mathcal{C} } ).</td>
</tr>
</tbody>
</table>
## Dictionary

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutation $\pi$</td>
<td>Permutation graph $G_\pi$</td>
</tr>
<tr>
<td>Containment $\pi &lt; \sigma$</td>
<td>Induced subgraph $G_\pi &lt;<em>{\text{ind}} G</em>\sigma$</td>
</tr>
<tr>
<td>Class $\mathcal{C}$</td>
<td>Class $G_{\mathcal{C}} = { G_\pi : \pi \in \mathcal{C} }$</td>
</tr>
<tr>
<td>$\text{Av}(321)$</td>
<td>Bipartite permutation graph</td>
</tr>
<tr>
<td>$\text{Av}(231)$</td>
<td>$\text{Free}(C_4, P_4)$</td>
</tr>
</tbody>
</table>
| $\text{Av}(312)$ | $\ldots$ }
### Dictionary

<table>
<thead>
<tr>
<th><strong>Permutations</strong></th>
<th><strong>Graphs</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutation $\pi$</td>
<td>Permutation graph $G_\pi$</td>
</tr>
<tr>
<td>Containment $\pi &lt; \sigma$</td>
<td>Induced subgraph $G_\pi &lt;<em>{\text{ind}} G</em>\sigma$</td>
</tr>
<tr>
<td>Class $\mathcal{C}$</td>
<td>Class $G_\mathcal{C} = { G_\pi : \pi \in \mathcal{C} }$</td>
</tr>
<tr>
<td>$\text{Av}(321)$</td>
<td>Bipartite permutation graph</td>
</tr>
<tr>
<td>$\text{Av}(231)$</td>
<td>Free($C_4, P_4$)</td>
</tr>
<tr>
<td>$\text{Av}(312)$</td>
<td>Free($\bullet$)</td>
</tr>
<tr>
<td>$\text{Av}(231, 312)$</td>
<td></td>
</tr>
</tbody>
</table>
## Dictionary

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permutation $\pi$</td>
<td>Permutation graph $G_\pi$</td>
</tr>
<tr>
<td>Containment $\pi &lt; \sigma$</td>
<td>Induced subgraph $G_\pi &lt;<em>{\text{ind}} G</em>\sigma$</td>
</tr>
<tr>
<td>Class $\mathcal{C}$</td>
<td>Class $G_\mathcal{C} = { G_\pi : \pi \in \mathcal{C} }$</td>
</tr>
<tr>
<td>$\text{Av}(321)$</td>
<td>Bipartite permutation graph</td>
</tr>
<tr>
<td>$\text{Av}(231)$</td>
<td>Free($C_4, P_4$)</td>
</tr>
<tr>
<td>$\text{Av}(312)$</td>
<td>— ” —</td>
</tr>
<tr>
<td>$\text{Av}(231, 312)$</td>
<td>Free($\bullet$)</td>
</tr>
<tr>
<td>$\text{Av}(2413, 3142)$ (separables)</td>
<td>Cographs: Free($P_4$)</td>
</tr>
<tr>
<td>$\text{Av}(3412, 2143)$ (skew-merged)</td>
<td>Split permutation graphs: Free($2K_2, C_4, C_5, S_3, \text{rising sun}, \text{net, rising sun}$)</td>
</tr>
</tbody>
</table>
Graphclasses.org tells me that:

\[
\text{Perm. graphs} = \text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36},
XF_1^{2n+3}, XF_2^{n+1}, XF_3^n, XF_4^n, XF_5^{2n+3}, XF_6^{2n+2},
+ \text{complements})
\]

N.B. (e.g.) $C_{n+4}$ are all the cycles of length $\geq 5$, so this is an infinite list.
Two interactions between permutations and graphs

1. Clique width
2. Labelled well-quasi-ordering
§1 Clique width
Build-a-graph

Set of labels $\Sigma$. You have 4 operations to build a labelled graph:

1. Create a new vertex with a label $i \in \Sigma$.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled $i$ to all labelled $j$, where $i, j \in \Sigma, i \neq j$.
4. Relabel every vertex labelled $i$ with $j$.

Example (Binary trees need at most 3 labels)
Build-a-graph

Set of labels $\Sigma$. You have 4 operations to build a labelled graph:

1. Create a new vertex with a label $i \in \Sigma$.
2. Disjoint union of two previously-constructed graphs.
3. Join all vertices labelled $i$ to all labelled $j$, where $i, j \in \Sigma, i \neq j$.
4. Relabel every vertex labelled $i$ with $j$.

- **Clique-width**, $cw(G) = \text{size of smallest } \Sigma \text{ needed to build } G$.
- If $H \leq_{\text{ind}} G$, then $cw(H) \leq cw(G)$.
- Clique-width of a class $\mathcal{C}$

\[
cw(\mathcal{C}) = \max_{G \in \mathcal{C}} cw(G)
\]

if this exists.
Motivation

Theorem (Courcelle, Makowsky and Rotics (2000))

If $cw(C) < \infty$, then any property expressible in monadic second-order (MSO$_1$) logic can be determined in polynomial time for $C$.

- MSO$_1$ includes many NP-hard algorithms: e.g. $k$-colouring ($k \geq 3$), graph connectivity, maximum independent set, …
- Generalises treewidth, critical to the proof of the Graph Minor Theorem (see next slide)
- Unlike treewidth, clique-width can cope with dense graphs
Diversion: treewidth, $tw(G)$

- $tw(G)$ measures ‘how like a tree’ $G$ is ($tw(G) = 1$ iff $G$ is a tree).
- Bounded treewidth $\implies$ all problems in MSO$_2$ in polynomial time.

**Theorem (Robertson and Seymour, 1986)**

For a minor-closed family of graphs $C$, $tw(C)$ bounded if and only if $C$ does not contain all planar graphs.

- Planar graphs are the unique “minimal” family for treewidth.

**Question**

Can we get a similar theorem for clique width?
Yes! For vertex-minors...

- Vertex-minor = induced subgraph + local complements

Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

Circle graph = intersection graph of chords
Yes! For vertex-minors...

- Vertex-minor = induced subgraph + local complements

Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

Circle graph = intersection graph of chords
Yes! For vertex-minors...

- Vertex-minor = induced subgraph + local complements

**Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))**

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

**Circle graph = intersection graph of chords**
Yes! For vertex-minors . . .

- Vertex-minor = induced subgraph + local complements

Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

Circle graph = intersection graph of chords
Yes! For vertex-minors...

- Vertex-minor = induced subgraph + local complements

**Theorem (Geelen, Kwon, McCarty, Wollan (announced 2018))**

A vertex-minor-closed downset of graphs has unbounded clique-width if and only if it contains every circle graph as a vertex-minor.

**Circle graph = intersection graph of chords**
Bounded clique-width

- Cographs, $\mathcal{C} = \text{Free}(P_4)$: $cw(\mathcal{C}) = 2$.
- $\mathcal{F} = \{\text{forests}\}$: $cw(\mathcal{F}) = 3$. 
### Bounded clique-width

- Cographs, $\mathcal{C} = \text{Free}(P_4)$: $cw(\mathcal{C}) = 2$.
- $\mathcal{F} = \{\text{forests}\}$: $cw(\mathcal{F}) = 3$.

### Unbounded clique-width

- Circle graphs
- Split permutation graphs
- Bipartite permutation graphs
- Any class with **superfactorial speed**
  ($\sim$ more than $n^{cn}$ labelled graphs of order $n$, for any $c$)
Clique width on graph classes

Bounded clique-width
- Cographs, $C = \text{Free}(P_4)$: $cw(C) = 2$.
- $\mathcal{F} = \{\text{forests}\}$: $cw(\mathcal{F}) = 3$.

Unbounded clique-width
- Circle graphs
- Split permutation graphs
- Bipartite permutation graphs
- Any class with superfactorial speed
  ($\sim$ more than $n^{cn}$ labelled graphs of order $n$, for any $c$)

Question
What are the minimal classes of graphs with unbounded clique-width?
Clique width on graph classes

Bounded clique-width

• Cographs, $\mathcal{C} = \text{Free}(P_4)$: $cw(\mathcal{C}) = 2$.
• $\mathcal{F} = \{\text{forests}\}$: $cw(\mathcal{F}) = 3$.

Unbounded clique-width

• Circle graphs
• Split permutation graphs $\leftarrow$ minimal!
• Bipartite permutation graphs $\leftarrow$ minimal!
• Any class with superfactorial speed
  ($\sim$ more than $n^{cn}$ labelled graphs of order $n$, for any $c$)

Question

What are the minimal classes of graphs with unbounded clique-width?
Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs
Permutations as circles

Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs
Permutations as circles

Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs

![Diagram showing Permutation graphs being a subset of Circle graphs](image)
Permutations as circles

Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs
Permutations as circles

Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs

\begin{center}
\begin{tikzpicture}
\draw[thick] (0,0) circle (2cm);
\draw (0,0) -- (1,0);
\draw (0,0) -- (0,1)
\end{tikzpicture}
\end{center}
Permutations as circles

Circle graphs are minimal with respect to vertex minor, but not for induced subgraphs.

Permutation graphs $\subset$ Circle graphs
Av(321) vs Bipartite permutation graphs

Theorem (Lozin, 2011)

Bipartite permutation graphs are a minimal class with unbounded clique-width.

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi = 321$</td>
<td>$G_{\pi}$ = [Graph Diagram]</td>
</tr>
</tbody>
</table>
Av(321) vs Bipartite permutation graphs

Theorem (Lozin, 2011)

Bipartite permutation graphs are a minimal class with unbounded clique-width.

Permutations

\[ \pi = 321 \]

Class: Av(321)

Graphs

\[ G_{\pi} \]

Bipartite permutation
Av(321) vs Bipartite permutation graphs

Theorem (Lozin, 2011)

Bipartite permutation graphs are a minimal class with unbounded clique-width.

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi = 321$</td>
<td>$G_\pi = \bullet$</td>
</tr>
<tr>
<td>Class:</td>
<td>Bipartite permutation</td>
</tr>
<tr>
<td>Av(321)</td>
<td></td>
</tr>
</tbody>
</table>
**Theorem (Lozin, 2011)**

Bipartite permutation graphs are a minimal class with unbounded clique-width.

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi = 321$</td>
<td>$G_{\pi} =$ Bipartite permutation</td>
</tr>
<tr>
<td>Class:</td>
<td>$\text{Av}(321)$</td>
</tr>
<tr>
<td>Structure:</td>
<td></td>
</tr>
</tbody>
</table>
**Split permutation graphs**

**Theorem (Atminas, B., Lozin, Stacho, 2018+)**

*Split permutation graphs are a minimal class with unbounded clique-width.*

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Merge of $1 \ldots k, j \ldots 1$</td>
<td>Indep set + clique</td>
</tr>
</tbody>
</table>
## Split permutation graphs

**Theorem (Atminas, B., Lozin, Stacho, 2018+)**

*Split permutation graphs are a minimal class with unbounded clique-width.*

<table>
<thead>
<tr>
<th>Permutations</th>
<th>Graphs</th>
</tr>
</thead>
<tbody>
<tr>
<td>Class:</td>
<td></td>
</tr>
<tr>
<td>Merge of $1 \ldots k, j \ldots 1$</td>
<td>Indep set + clique</td>
</tr>
<tr>
<td>Av(2143, 3412)</td>
<td>Split permutation</td>
</tr>
</tbody>
</table>
Split permutation graphs

Theorem (Atminas, B., Lozin, Stacho, 2018+)

Split permutation graphs are a minimal class with unbounded clique-width.

Permutations

Class: \( \text{Av}(2143, 3412) \)

Structure:

Graphs

Indep set + clique

Split permutation
Split permutation graphs

Theorem (Atminas, B., Lozin, Stacho, 2018+)

Split permutation graphs are a minimal class with unbounded clique-width.

Class:
Av(2143, 3412)

Permutations
Merge of 1 \ldots k, j \ldots 1
Graphs
Indep set + clique
Split permutation
More minimal classes?

- Permutation class structure is a long ‘path’:

- Could find minimal classes of permutation graphs.
- Carry out local complementation to make other (non-permutation) graph classes.

Corollary (to Geelen, Kwon, McCarty, Wollan)

Every minimal class of unbounded clique width is a subclass of circle graphs.

Question (possibly naive)

Are all these classes related to each other by local complementation?
§2 Labelled well-quasi-ordering
Infinite labelled antichains

- **Antichain**: set of pairwise incomparable graphs/permutations

The set of cycles forms an infinite antichain

\[ \triangle, \quad \square, \quad \pentagon, \quad \hexagon, \quad \ldots \]
Infinite labelled antichains

- **Antichain**: set of pairwise incomparable graphs/permutations

The set of cycles forms an infinite antichain

Paths form a *labelled* infinite antichain
Infinite labelled antichains

- **Antichain**: set of pairwise incomparable graphs/permutations

The set of cycles forms an infinite antichain

Paths form a *labelled* infinite antichain

Increasing oscillations/Gollan permutations too...
Infinite labelled antichains

- **Antichain**: set of pairwise incomparable graphs/permutations

The set of cycles forms an infinite antichain

Paths form a *labelled* infinite antichain

Increasing oscillations/Gollan permutations too...
No labelled antichains

- well-quasi-order (WQO): no infinite antichain.
- Labelled well-quasi-order (LWQO): no infinite labelled antichain.

**Theorem (Pouzet, 1972)**

Every LWQO class (of graphs, permutations, anything) is finitely based.

**Conjecture (Korpelainen, Lozin & Razgon, 2013; Atminas & Lozin, 2015)**

Every finitely based WQO graph class must also be LWQO.
Conjecture (KLR, 2013; AL, 2015)

*Every finitely based WQO graph class must also be LWQO.*

If a graph contains long paths, then it contains

\[ \begin{array}{c}
\circ \bullet \circ \quad \circ \bullet \circ \bullet \circ \quad \circ \bullet \circ \bullet \circ \bullet \circ \quad \circ \bullet \circ \bullet \circ \bullet \circ \bullet \circ \quad \cdots
\end{array} \]

…so is not LWQO.
Conjecture (KLR, 2013; AL, 2015)

*Every finitely based WQO graph class must also be LWQO.*

If a graph contains long paths, then it contains

...so is not LWQO.

But then, you can’t avoid

...unless they are all in the basis.
‘Obviously’ wrong for permutations

Increasing oscillations again

Proposition

The smallest class containing the increasing oscillations is $Av(321, 2341, 3412, 4123)$ and is WQO (but not LWQO).
'Obviously' wrong for permutations

Increasing oscillations again

Proposition

The smallest class containing the increasing oscillations is \( Av(321, 2341, 3412, 4123) \) and is WQO (but not LWQO).

But...

As a graph class, \( C_n \) is a basis element for \( n \geq 5 \).
\( \Rightarrow \) not a counterexample.
Another permutation example

**Proposition (B., Engen, Vatter, 2018+)**

$Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162)$ is another WQO-but-not-LWQO class.

**Here’s the labelled antichain**

...
Another permutation example

Proposition (B., Engen, Vatter, 2018+)

$Av(2143, 2413, 3412, 314562, 412563, 415632, 431562, 512364, 512643, 516432, 541263, 541632, 543162)$ is another WQO-but-not-LWQO class.

Here’s the labelled antichain

Corollary

The class $Free(2K_2, C_4, C_5, net, co-net, risingsun, co-rising sun, H, \overline{H}, cross,
cross, X_{168}, \overline{X_{168}}, X_{160})$, is WQO but not LWQO.
Conjecture (Daligault, Rao, Thomassé, 2010)

If \( C \) is labelled well-quasi-ordered, then \( C \) has bounded clique-width.

N.B.

WQO does not imply bounded clique width (Lozin, Razgon, Zamaraev, 2018).
Thanks!