Well-quasi-ordering permutations

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Joint work with Vince Vatter (U. Florida)

Scottish Combinatorics Meeting, University of Strathclyde, 23 May 2023
Embedding graphs

Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?

Yes!
Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?

Yes!
Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?

No!
Embedding graphs

Can I delete vertices (and incident edges) from the graph on the right to produce the graph on the left?

No!
Is any graph in the following (infinite) list an induced subgraph of another?
Is any graph in the following (infinite) list an induced subgraph of another?

No!
Can I delete points from the picture on the right, and rescale, to form the picture on the left?
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No!
No permutation in the following list embeds in any other
**Infinite antichain**: An infinite set of combinatorial structures such that no one embeds in another.
A library of infinite antichains of permutations
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§1 Permutation containment
• Think of the $n$ entries of $\pi = \pi(1) \cdots \pi(n)$ as vertices
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• *Containment* ordering: ‘Delete entries, and rescale’
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• Formally: $\sigma \leq \pi$ if $\pi$ has a subsequence with the same relative ordering as $\sigma$. 
• Think of the $n$ entries of $\pi = \pi(1) \cdots \pi(n)$ as vertices

• **Containment** ordering: ‘Delete entries, and rescale’

• Formally: $\sigma \leq \pi$ if $\pi$ has a subsequence with the same relative ordering as $\sigma$.

• If $\sigma \not\leq \pi$, then $\pi$ avoids $\sigma$. 
Inversion graph $G_\pi$ of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$
- Edges: $a \sim b$ if $a < b$ and $\pi(b) < \pi(a)$

(edges = inversions)
Inversion graph $G_\pi$ of $\pi = \pi(1) \cdots \pi(n)$:

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Inversion graph $G_\pi$ of $\pi = \pi(1) \cdots \pi(n)$:

- Vertices $= \{1, 2, \ldots, n\}$
- Edges: $a \sim b$ if $a < b$ and $\pi(b) < \pi(a)$ (edges $=$ inversions)

Induced substructure preserved: $\sigma \leq \pi$ implies $G_\sigma \leq \text{ind } G_\pi$
Permutations to graphs is many-to-one

$\sigma \leq \pi$ implies $G_\sigma \leq_{\text{ind}} G_\pi$ but:

$G_{2413} \cong G_{3142} \cong \cdots$ even though $2413 \neq 3142$. 
Hereditary classes

Set $\mathcal{C}$ of graphs/permutations is hereditary if $A \in \mathcal{C}$ and $B$ is an induced substructure of $A$, then $B \in \mathcal{C}$. ('class')

Every hereditary class has a unique set of minimal forbidden elements: the smallest things that are ‘not in the class’. ('basis')
Hereditary classes

Set \( C \) of graphs/permutations is hereditary if
\[ A \in C \text{ and } B \text{ is an induced substructure of } A, \text{ then } B \in C. \] (‘class’)

Every hereditary class has a unique set of minimal forbidden elements:
the smallest things that are ‘not in the class’. (‘basis’)

Some graph classes

<table>
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<tr>
<th>Class ( C = \text{Free}(\mathcal{B}) )</th>
<th>Basis ( \mathcal{B} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Empty graphs (no edges)</td>
<td>{}</td>
</tr>
<tr>
<td>Forests</td>
<td>{\text{triangle, square, circle, ...}}</td>
</tr>
<tr>
<td>Bipartite graphs</td>
<td>{\text{triangle, circle, square, ...}}</td>
</tr>
<tr>
<td>Inversion graphs</td>
<td>\text{Free}(C_{n+4}, T_2, X_2, X_3, X_{30}, X_{31}, X_{32}, X_{33}, X_{34}, X_{36}, XF_1^{2n+3}, XF_2^{n+1}, XF_3^n, XF_4^n, XF_5^{2n+3}, XF_6^{2n+2}, +complements)</td>
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<td>(Gallai 1967)</td>
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Hereditary classes

Set $\mathcal{C}$ of graphs/permutations is **hereditary** if $A \in \mathcal{C}$ and $B$ is an induced substructure of $A$, then $B \in \mathcal{C}$. ('class')

Every hereditary class has a unique set of **minimal forbidden elements**: the smallest things that are ‘not in the class’. ('basis')

### Some permutation classes

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<th>Class $\mathcal{C} = \text{Av}(\mathfrak{B})$</th>
<th>Basis $\mathfrak{B}$</th>
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<tr>
<td>${1, 12, 123, \ldots }$</td>
<td>${21}$</td>
</tr>
<tr>
<td>Union of 2 increases</td>
<td>${321}$</td>
</tr>
<tr>
<td>‘Stack sortable’</td>
<td>${231}$</td>
</tr>
<tr>
<td>‘2-stack-sortable’</td>
<td>Infinite (Murphy 2003)</td>
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§2 Counting classes
Av(21) = \{1, 12, 123, \ldots\} has 1 permutation of each length.
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<th>Av(21) = {1, 12, 123, \ldots} has 1 permutation of each length.</th>
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![Graph for Av(21)](image1)

![Graph for Av(231)](image2)
\( \text{Av}(21) = \{1, 12, 123, \ldots \} \) has 1 permutation of each length.

\( \text{Av}(231) \) has 1, 2, 5, 14, 42, \ldots \) of lengths \( n = 1, 2, 3, 4, 5, \ldots \).

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\( \text{Av}(321) \) has 1, 2, 5, 14, 42, \ldots of lengths \( n = 1, 2, 3, 4, 5, \ldots \).

\( \text{Av}(132, 321) \) has \( \binom{n}{2} + 1 \) of length \( n \).
Typical questions in Permutation Patterns

For a permutation class $\mathcal{C}$:

- What is the generating function? (e.g. rational, algebraic, $D$-finite)

$$f_\mathcal{C}(z) = \sum_{\pi \in \mathcal{C}} z^{\vert \pi \vert} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n, \text{ where } \mathcal{C}_n = \{ \pi \in \mathcal{C} : \vert \pi \vert = n \}.$$

- What is the growth rate?

$$\text{gr}(\mathcal{C}) = \limsup_{n \to \infty} n \sqrt[n]{|\mathcal{C}_n|}, \text{ which exists by Marcus \& Tardos (2004).}$$

- What is the basis? (Is $\mathcal{C}$ finitely based?)

- What do the permutations ‘look like’?
\[ f_{Av(21)} = \frac{1}{1 - z}; \quad gr = 1. \]
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\[ f_{Av(231)} = \frac{1 - \sqrt{1 - 4z}}{2z}; \quad gr = 4. \]
\[ f_{Av(21)} = \frac{1}{1-z}; \quad \text{gr} = 1. \]

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\[ f_{\text{Av}(321)} = \frac{1 - \sqrt{1 - 4z}}{2z}; \quad \text{gr} = 4. \]

\[ f_{\text{Av}(132,123)} = \frac{1 - z + z^2}{(1-z)^3}; \quad \text{gr} = 1. \]
‘Tame’ structure tends to give a ‘tame’ generating function.
Theorem (Albert, B., 2014)

The permutation class that determine Schubert varieties defined by inclusions $\text{Av}(4231, 31524, 42513, 351624)$ has generating function

$$\frac{1 - 3z - 2z^2 - (1 - z - 2z^2)\sqrt{1 - 4z}}{1 - 3z - (1 - z + 2z^2)\sqrt{1 - 4z}}.$$
‘Tame’ structure tends to give a ‘tame’ generating function.

but not vice versa...
**Theorem (Albert, B., Vatter, 2013)**

*Every proper permutation class \( C \) is contained in a permutation class with a rational generating function.*

**‘Proof’**.

Make an enormous infinite antichain
Theorem (Albert, B., Vatter, 2013)

Every proper permutation class \( \mathcal{C} \) is contained in a permutation class with a rational generating function.

‘Proof’.

Make an enormous infinite antichain \( \mathcal{A} \)

such that \( \text{Av}(\mathcal{A}) \cup \mathcal{A} \) has a rational generating function.
Theorem (Albert, B., Vatter, 2013)

**Every proper permutation class** $\mathcal{C}$ **is contained in a permutation class with a rational generating function.**

*Proof*.

Make an enormous infinite antichain $\mathcal{A}$ such that $\text{Av}(\mathcal{A}) \cup \mathcal{A}$ has a rational generating function. Union this with $\mathcal{C}$, and remove enough antichain elements of each length to preserve rationality.
‘Tame’ structure tends to give a ‘tame’ generating function.

It is tempting to generalise...
Conjecture (Noonan, Zeilberger, 1996)

Every finitely based class has a D-finite generating function.
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So ‘finitely based’ isn’t universally tame. Nevertheless…
Conjecture (Noonan, Zeilberger, 1996)

*Every finitely based class has a D-finite generating function.*

Conjecture (Zeilberger, 2005)

*Noonan-Zeilberger is false.*

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*Zeilberger is right: Noonan-Zeilberger is false.*

So ‘finitely based’ isn’t universally tame. Nevertheless…

Conjecture

*Every finitely based class with growth rate $< 4$ has a rational generating function.*
‘Tame’ structure tends to give a ‘tame’ generating function.

Is there something general we can say here?
§3 Well-quasi-ordering
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<td><strong>Growth rate</strong></td>
<td>4</td>
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<td><strong>Basis</strong></td>
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‘Look like’
What about subclasses of $\text{Av}(231), \text{Av}(321)$?

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<td>Infinite antichains?</td>
<td>No</td>
<td>Yes: ( \cdots )</td>
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A permutation class is **well-quasi-ordered (WQO)** if it contains no infinite antichains.
A permutation class is well-quasi-ordered (WQO) if it contains no infinite antichains.

A strong indicator of ‘tameness’, for example, even though $\text{Av}(321)$ is not WQO:

**Theorem (Albert, B., Ruškuc, Vatter, 2019)**

Every WQO or finitely based subclass of $\text{Av}(321)$ has a rational generating function.
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*Every WQO permutation class has an algebraic generating function.*
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**Conjecture (Vatter, 2015)**

Every WQO permutation class has an algebraic generating function.

This also turned out to be a generalisation too far…
Prouhet-Thue-Morse:

\[0110 1001 1001 0110 1001 0110 0110 1001 1001 0110 0110 1001 \ldots\]

is a uniformly recurrent sequence

Infinite binary sequence \(s \rightarrow\) infinite permutation \(\pi_s : \mathbb{N} \rightarrow \mathbb{Z}\)

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Infinite binary sequence $s \rightarrow$ infinite permutation $\pi_s : \mathbb{N} \rightarrow \mathbb{Z}$
Prouhet-Thue-Morse:
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Infinite binary sequence $s \rightarrow$ infinite permutation $\pi_s : \mathbb{N} \rightarrow \mathbb{Z}$
Prouhet-Thue-Morse: 0110 1001 1001 0110 1001 0110 1001 1001 0110 0110 1001

Infinite binary sequence $s \rightarrow$ infinite permutation $\pi_s : \mathbb{N} \rightarrow \mathbb{Z}$
Infinite binary sequence $s \longrightarrow$ infinite permutation $\pi_s : \mathbb{N} \to \mathbb{Z}$
Prouhet-Thue-Morse: $01101001100101101001011001101001011001101001101001011001$. . .

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Infinite binary sequence $s \rightarrow$ infinite permutation $\pi_s : \mathbb{N} \rightarrow \mathbb{Z}$
Prouhet-Thue-Morse:
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class $\mathcal{C}_s = \{(\text{finite}) \text{ permutations} \pi \leq \pi_s\}$
Prouhet-Thue-Morse:
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Infinite binary sequence $s \rightarrow$ infinite permutation $\pi_s : \mathbb{N} \rightarrow \mathbb{Z}$

class $\mathcal{C}_s = \{(\text{finite}) \text{ permutations } \pi \leq \pi_s\}$

If $s$ is uniformly recurrent, then $\mathcal{C}_s$ is wqo.
Sequence construction (generalising Prouhet-Thue-Morse) from Maurice Pouzet’s 1978 thesis \(\implies\) uncountably many WQO classes with \textit{different} generating functions.

Too many generating functions for all of them to be algebraic.

Infinite binary sequence \(s\) \(\rightarrow\) infinite permutation \(\pi_s : \mathbb{N} \rightarrow \mathbb{Z}\)

class \(C_s = \{(\text{finite}) \text{ permutations} \pi \leq \pi_s\}\)

If \(s\) is \textit{uniformly recurrent}, then \(C_s\) is wqo.
Sequence construction (generalising Prouhet-Thue-Morse) from Maurice Pouzet’s 1978 thesis \(\implies\) uncountably many WQO classes with different generating functions.

Too many generating functions for all of them to be algebraic.

\textbf{Theorem (B., Vatter, 2023+)}

There are uncountably many distinct enumerations of WQO permutation classes.

Hence, not all WQO classes have algebraic (or D-finite) generating functions.

Infinite binary sequence \(s\) \(\longrightarrow\) infinite permutation \(\pi_s : \mathbb{N} \to \mathbb{Z}\)

\[\text{class } \mathcal{C}_s = \{(\text{finite}) \text{ permutations } \pi \leq \pi_s\}\]

If \(s\) is uniformly recurrent, then \(\mathcal{C}_s\) is wqo.
§4 Labelled WQO
A regular infinite antichain:

A labelled infinite antichain:

Labels can be (partially) ordered (e.g. $\preceq$): embedding must respect the label ordering.
A regular infinite antichain:

A labelled infinite antichain:

Labels can be (partially) ordered (e.g. \( \bullet \preceq \circ \)): embedding must respect the label ordering.
 Labelled WQO

A class is **labelled well-quasi-ordered** (LWQO) if we cannot construct a labelled infinite antichain, no matter the set of labels.†

† Includes infinite sets of labels, but they **must** be WQO.

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**Theorem (After Pouzet, 1972)**

*LWQO (permutation) classes must be finitely based.*

**Corollary**

*There are only countably many LWQO permutation classes.*
‘Tame’ structure tends to give a ‘tame’ generating function.

Does LWQO guarantee tame enumeration?
§5 Permutations & inversion graphs
Does WQO translate?

Recall: \( \sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi \).

Thus \( \mathcal{C} (\text{L})\text{WQO} \Rightarrow G_\mathcal{C} (\text{L})\text{WQO} \).

**Question**

*If \( \mathcal{C} \) is a permutation class such that \( G_\mathcal{C} \) is WQO, must \( \mathcal{C} \) be WQO?*
Does WQO translate?

Recall: $\sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi$. Thus $\mathcal{C}(L)\text{WQO} \Rightarrow G_\mathcal{C}(L)\text{WQO}$.

**Question**

If $\mathcal{C}$ is a permutation class such that $G_\mathcal{C}$ is WQO, must $\mathcal{C}$ be WQO?

This question seems to be very difficult. Here is a permutation antichain which turns into a chain of graphs:

Note that $G_{231} \cong G_{312} \cong \cdot$
Does WQO translate?

Recall: \( \sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi \). Thus \( C \subseteq (L)\text{WQO} \Rightarrow G_C \supseteq (L)\text{WQO} \).

**Question**

*If \( C \) is a permutation class such that \( G_C \) is WQO, must \( C \) be WQO?*

This question seems to be very difficult. Here is a permutation antichain which turns into a chain of graphs:

Note that \( G_{231} \cong G_{312} \cong \triangleleft \).
Does WQO translate?

Recall: $\sigma \leq \pi \Rightarrow G_\sigma \leq_{\text{ind}} G_\pi$. Thus $\mathcal{C} \text{(L)WQO} \Rightarrow G_\mathcal{C} \text{(L)WQO}$.

Question

If $\mathcal{C}$ is a permutation class such that $G_\mathcal{C}$ is WQO, must $\mathcal{C}$ be WQO?

Theorem (B., Vatter, 2022)

Let $\mathcal{C}$ be a permutation class. $\mathcal{C}$ is LWQO if and only if $G_\mathcal{C}$ is LWQO.
Thanks!