Simple Extensions of Relational Structures

R.L.F. Brignall[1]
joint work with Nik Ruškuc[2] and Vincent Vatter

[1]Department of Mathematics
University of Bristol

[2]School of Mathematics and Statistics
University of St Andrews

Friday 7th December, 2007

Introduction

Outline

Sets and Relations

- A relational structure: a set of points, and a set of relations on these points.

Sets and Relations

- A relational structure: a set of points, and a set of relations on these points.
- The ground set, A.

Sets and Relations

- A relational structure: a set of points, and a set of relations on these points.
- The ground set, A.
- A k-ary relation R – a subset of A^k.

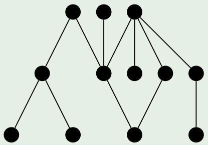
Sets and Relations

- A relational structure: a set of points, and a set of relations on these points.
- The ground set, A.
- A k-ary relation R – a subset of A^k.
- Binary relations come in many different flavours – linear, transitive, symmetric,...

Posets

- Poset — a relational structure on a binary reflexive antisymmetric transitive relation.

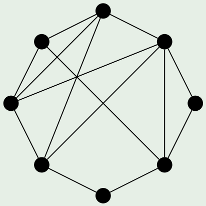
Example



Graphs

- Graph — a relational structure on a single binary symmetric relation.

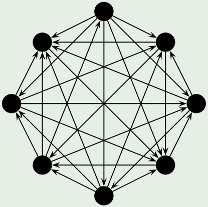
Example



Tournaments

- Tournament — a complete oriented graph.

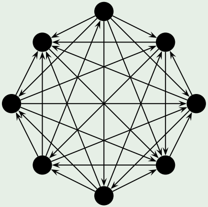
Example



Tournaments

- Tournament — a complete oriented graph.
- As a relational structure, it is a single trichotomous binary relation — $x \to y$, $y \to x$ or $x = y$.

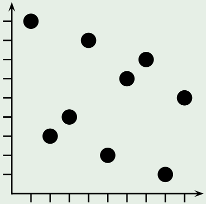
Example



Permutations

- Permutation of length n — a structure on two linear relations.

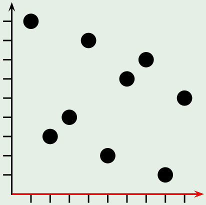
Example



Permutations

- Permutation of length n — a structure on two linear relations.

Example

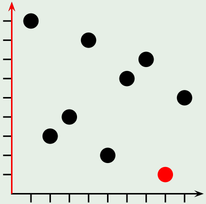


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

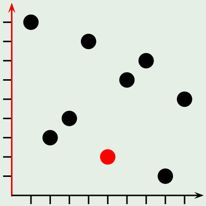


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- 8

Permutations

- Permutation of length n — a structure on two linear relations.

Example

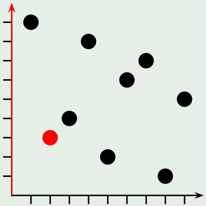


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

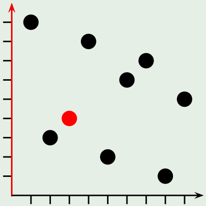


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

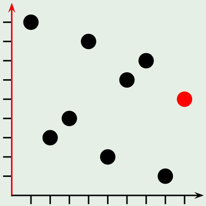


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$.
- $8 \prec 5 \prec 2 \prec 3$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

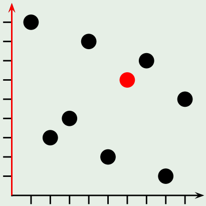


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

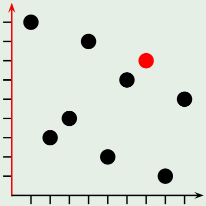


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

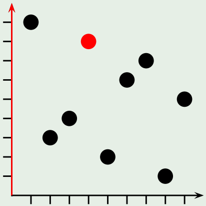


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7$

Permutations

- Permutation of length n — a structure on two linear relations.

Example

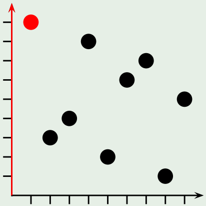


- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7 \prec 4$

Permutations

- Permutation of length n — a structure on two linear relations.

Example



- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7 \prec 4 \prec 1$

Intervals

- An interval: set of points which “look” at every other point in the same way.

Intervals

- An interval: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, modules...

Intervals

- An interval: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, modules...
- A structure is simple if there are no proper intervals.

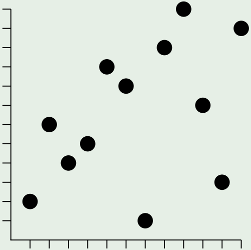
Intervals

- An interval: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, modules...
- A structure is simple if there are no proper intervals.
- Synonyms: Indecomposable, prime...

Permutations

- Permutation π.

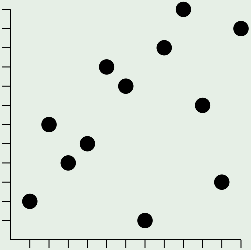
Example



Permutations

- Permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

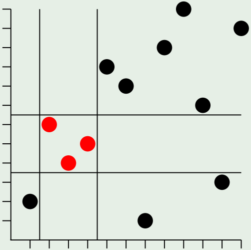
Example



Permutations

- Permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

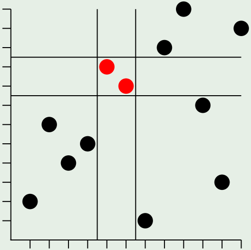
Example



Permutations

- Permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

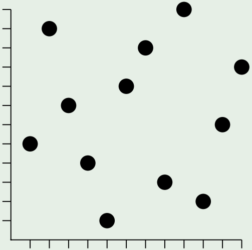
Example



Permutations

- Permutation π.
- Simple permutations — only intervals are singletons and the whole thing.

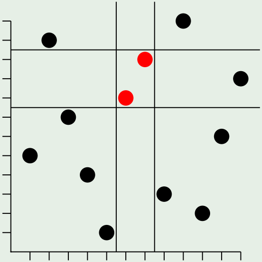
Example



Permutations

- Permutation π.
- Simple permutations — only intervals are singletons and the whole thing.

Example



Simplicity in Graphs

- Simple graph?

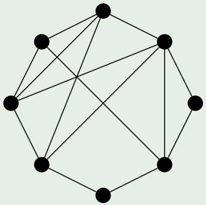
Simplicity in Graphs

- Simple graph? Well, rather an indecomposable graph.

Simplicity in Graphs

- Simple graph? Well, rather an indecomposable graph.

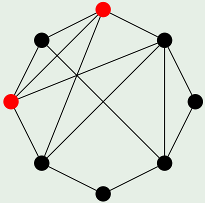
Example



Simplicity in Graphs

- Simple graph? Well, rather an indecomposable graph.

Example



- Same (outside) neighbourhood = interval.

The Question

Question

How many additional points are needed to extend a given relational structure to a simple one?

The Question

Question

How many additional points are needed to extend a given relational structure to a simple one?

- Call this a simple extension.

History: Tournament Extensions

Theorem (Erdős, Fried, Hajnal and Milner, 1972)

Every tournament has a simple extension with at most two additional vertices.

History: Tournament Extensions

Theorem (Erdős, Fried, Hajnal and Milner, 1972)

Every tournament has a simple extension with at most two additional vertices.

- N.B. this means arbitrary cardinality.

History: Tournament Extensions

Theorem (Erdős, Fried, Hajnal and Milner, 1972)

Every tournament has a simple extension with at most two additional vertices.

- N.B. this means arbitrary cardinality.

Theorem (Erdős, Hajnal and Milner, 1972)

A tournament T has a one-point simple extension unless $|T| = 3$ or T has an odd number of vertices and is transitive.

Some Thoughts

- 2 extra points isn't always going to be enough (think of K_n)

Some Thoughts

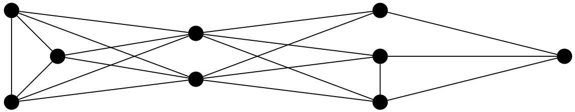
- 2 extra points isn't always going to be enough (think of K_n)
- Have to consider different binary structures separately...

Some Thoughts

- 2 extra points isn't always going to be enough (think of K_n)
- Have to consider different binary structures separately...
- ... *but* the approach is going to be similar.

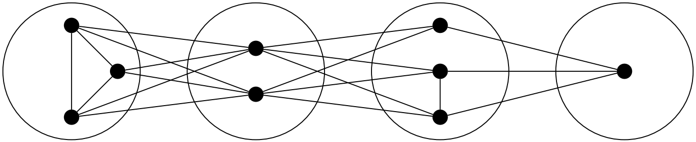
Outline

Substitution Decomposition



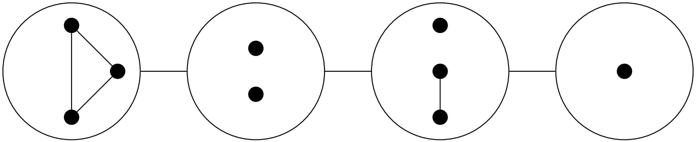
- Take any graph (more generally: relational structure).

Substitution Decomposition



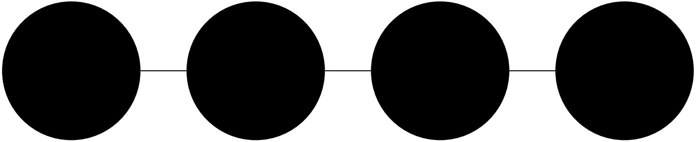
- Find the maximal proper intervals (N.B. they don't intersect).

Substitution Decomposition



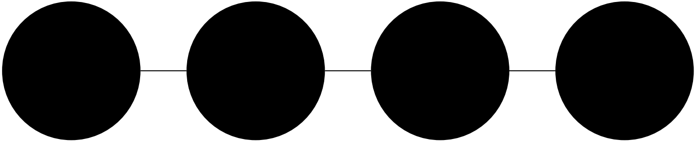
- Find the maximal proper intervals (N.B. they don't intersect).

Substitution Decomposition



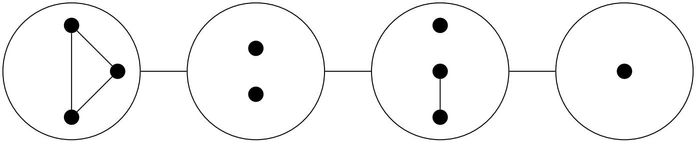
- Replace each interval with a single point.

Substitution Decomposition



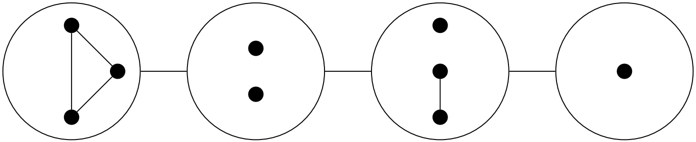
- Replace each interval with a single point.
- Have the skeleton — P_4 — which is indecomposable.

Substitution Decomposition



- This is the substitution decomposition
(Also called modular decomposition, disjunctive decomposition, X-join).

Substitution Decomposition



- This is the substitution decomposition (Also called modular decomposition, disjunctive decomposition, X-join).
- Unique unless skeleton is K_n or $\overline{K_n}$.

The Approach (for Binary Structures)

- Induction using the substitution decomposition.

The Approach (for Binary Structures)

- Induction using the substitution decomposition.
- Non-unique cases handled separately.

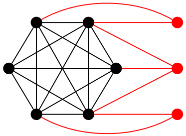
The Approach (for Binary Structures)

- Induction using the substitution decomposition.
- Non-unique cases handled separately.
- Bound obtained tends to be tight on the non-unique cases.

Graphs

Theorem (Sumner, 1971)

K_n has a simple extension with $\lceil \log_2(n+1) \rceil$ additional vertices.



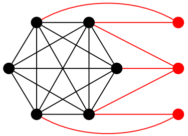
Graphs

Theorem (Sumner, 1971)

K_n has a simple extension with $\lceil \log_2(n+1) \rceil$ additional vertices.

Theorem

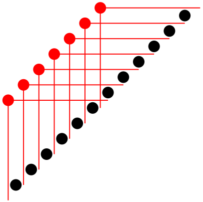
A graph on n vertices has a simple extension requiring at most $\lceil \log_2(n+1) \rceil$ additional vertices.



Permutations

Theorem

A permutation on n points has a simple extension requiring at most $\left\lceil \frac{n+1}{2} \right\rceil$ *additional points.*



Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.

Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs.

Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs. Get $\lceil \log_2(n+1) \rceil$.

Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs. Get $\lceil \log_2(n+1) \rceil$.
- Linear orders behave like permutations.

Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs. Get $\lceil \log_2(n+1) \rceil$.
- Linear orders behave like permutations. Get $\lceil (n+1)/2 \rceil$.

Posets: A Graph-Permutation Mix

- Two (different) bad cases: antichains and linear orders.
- Antichains behave like graphs. Get $\lceil \log_2(n+1) \rceil$.
- Linear orders behave like permutations. Get $\lceil (n+1)/2 \rceil$.

Theorem

A poset with n elements has a simple extension requiring at most $\left\lceil \frac{n+1}{2} \right\rceil$ *additional elements.*

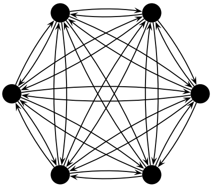
Outline

Many Cases

- Digraphs — the most general type of binary structure.

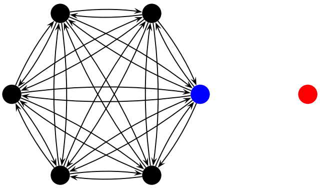
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs.



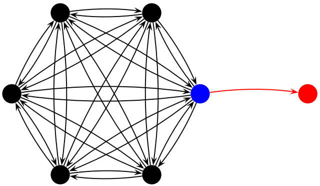
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs.



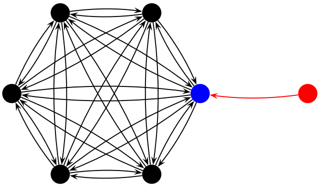
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs.



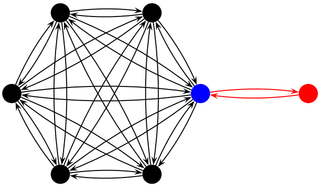
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs.



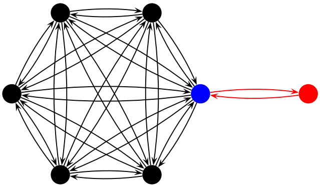
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs.



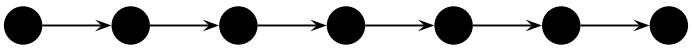
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs — get $\lceil \log_4(n+1) \rceil$.



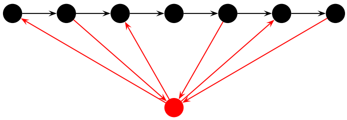
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs — get $\lceil \log_4(n+1) \rceil$.
- Linear orders.



Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs — get $\lceil \log_4(n+1) \rceil$.
- Linear orders — one-point extension suffices.



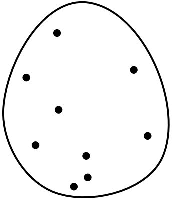
Many Cases

- Digraphs — the most general type of binary structure.
- Complete or empty digraphs — get $\lceil \log_4(n+1) \rceil$.
- Linear orders — one-point extension suffices.

Theorem

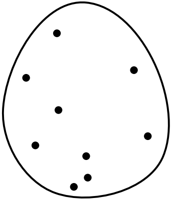
A digraph with n elements has a simple extension requiring at most $\lceil \log_4(n+1) \rceil$ additional elements.

k-ary Relations



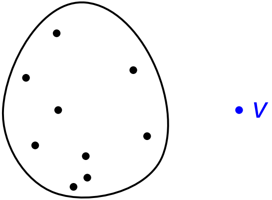
- Structure contains a k-ary relation, $k \geq 3$.

k-ary Relations



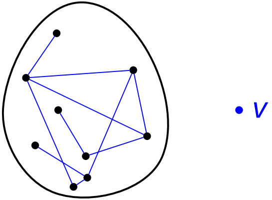
- Structure contains a k-ary relation, $k \geq 3$.
- e.g. terms of a 3-ary relation look like $(\_, \_, \_)$.

k-ary Relations



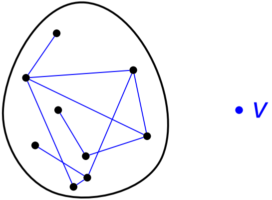
- Structure contains a k-ary relation, $k \geq 3$.
- e.g. terms of a 3-ary relation look like $(\_, \_, \_)$.
- Add one new point, v, say.

k-ary Relations



- Structure contains a k-ary relation, $k \geq 3$.
- e.g. terms of a 3-ary relation look like $(\_, \_, \_)$.
- Add one new point, v, say.
- Add relations $(v, \_, \_)$ so last two coordinates form a simple structure.

k-ary Relations



Theorem

Every relational structure which has an arbitrary k-ary relation with $k \geq 3$ has a one-point simple extension.