From permutations to graphs
well-quasi-ordering and infinite antichains

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Joint work with Atminas, Korpelainen, Lozin and Vatter

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Orderings on Structures

- Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, ...
Orderings on Structures

• Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, ...
• Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment,
Orderings on Structures

• Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, …
• Give your family an ordering. E.g. graph minor, induced subgraph, permutation containment, …
• Does your ordering contain infinite antichains? i.e. an infinite set of pairwise incomparable elements.

Example ((Induced) subgraph antichains)

Cycles:

```
  △   □   五金   七边形   …
```

“Split end” graphs:

```
  △   □   五金   七边形   …
```
When are there antichains?

No infinite antichains $=$ well-quasi-ordered.

- **Words** over a finite alphabet with subword ordering [Higman, 1952].
- **Trees** ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- **Graphs closed under minors** [Robertson and Seymour, 1983—2004].

Infinite antichains.

- Graphs closed under **induced subgraphs** (or merely subgraphs).
- Permutations closed under **containment**.
- Tournaments, digraphs, posets, ... with their natural **induced substructure** ordering.
Algorithms inside well-quasi-ordered sets

- Polynomial-time recognition: is one graph a minor of another?
- Fixed-parameter tractability: e.g. graphs with vertex cover at most $k$ can be recognised in polynomial time.

Miscellany

- Well-quasi-order = nice structure. Useful for other problems (e.g. enumeration)
- Connections with logic: Kruskal’s Tree Theorem is unproveable in Peano arithmetic [Friedman, 2002]
- Antichains are pretty! (See later)
- It is fun [Kříž and Thomas, 1990]
- Because it’s there. [Mallory]
Formal definition

• Quasi order: reflexive transitive relation.
• Partial order: quasi order + asymmetric.

Definition

Let \((S, \leq)\) be a quasi-ordered (or partially-ordered) set. Then \(S\) is said to be well quasi ordered (wqo) under \(\leq\) if it

• is well-founded (no infinite descending chain), and
• contains no infinite antichain (set of pairwise incomparable elements).

• Well founded: usually trivial for finite combinatorial objects. This is all about the antichains.
My objects aren’t wqo, what can I do?

- Don’t panic! Maybe you could restrict to a subcollection?

**Example: Cographs as induced subgraphs**

\[ \text{Cographs} = \text{graphs containing no induced } P_4 \]

\[ = \text{closure of } K_1 \text{ under complementation and disjoint union.} \]

- Cographs are well-quasi-ordered. [Damaschke, 1990]

- Learn to stop worrying and love the antichains! [sorry, Kubrick]
**Question**

In your favourite ordering, which downsets contain infinite antichains?

- Downset (or hereditary property, or class): set $C$ of objects such that
  
  $G \in C$ and $H \leq G$ implies $H \in C$.

**Examples**

- Triangle-free graphs: downset under (induced) subgraphs. Not wqo.
- Cographs: downset under induced subgraphs. Wqo.
- Planar graphs: downset under graph minor. Wqo.
- Words over $\{0, 1\}$ with no ‘00’ factor: downset under factor order. Not wqo: 010, 0110, 01110, 011110, …
Minimal forbidden elements

- Downsets often defined by the minimal forbidden elements.

**Examples**

- Triangle-free graphs: $K_3$ free as (induced) subgraph.
- Cographs: Free($P_4$).
- Planar graphs: $\{K_5, K_{3,3}\}$-minor free graphs [Wagner’s Theorem]
- Pattern-avoiding permutations: Av(321) (see later).

- Confusingly, the set of minimal forbidden elements is an antichain!

- Graph Minor Theorem $\Rightarrow$ every minor-closed class has finitely many forbidden elements.
Decision procedures

Question

In your favourite ordering, which downsets contain infinite antichains?

Known decision procedures

• **Graph minors**: no antichains anywhere!
• **Subgraph order**: a downset is wqo if and only if it contains neither
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  \boxdot
  \end{array}
  \, \cdots
  \begin{array}{c}
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  \prec
  \end{array}
  \cdots [Ding, 1992]
• **Factor order**: downsets of words over a finite alphabet [Atminas, Lozin & Moshkov, 2013]

Theorem (Cherlin & Latka, 2000)

Any downset with \(k\) minimal forbidden elements is wqo iff it doesn’t contain any of the infinite antichains in a finite collection \(\Lambda_k\).
Plan for the rest of today

Ordering of the day
Induced subgraph ordering, $H \leq_{\text{ind}} G$.

Question
For which $m, n$ is the following true?

The set of permutation graphs with no induced $P_m$ or $K_n$ is wqo.

We’ll:
- Build some antichains;
- Find structure to prove wqo.

Motivation?
- The ‘right’ level of difficulty: Interestingly complex, but tractable.
- Demonstration of some recently-developed structural theory.
- Expansion of the graph $\longleftrightarrow$ permutation interplay.
Forbidding paths and cliques

- $m$: size of forbidden path
- $n$: size of forbidden clique

- ● = Graphs wqo
- ○ = Permutation graphs wqo, graphs not wqo
- ○○ = Permutation graphs not wqo
These classes are trivially wqo.
Cographs are wqo [Damaschke, 1990]
Permutation graphs

$P_6, K_3$-free graphs are wqo [Atminas and Lozin, 2014]
Permutation graphs

$P_5, K_4$-free graphs are not wqo [Korpelainen and Lozin, 2011]
Permutation graphs

$P_7, K_3$-free graphs are not wqo [Korpelainen and Lozin, 2011b]
Forbidding paths and cliques

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Permutation graphs

• Permutation $\pi = \pi(1) \cdots \pi(n)$

• Make a graph $G_\pi$: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.

• Note: $n \cdots 21$ becomes $K_n$. 
Permutation graphs

- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph $G_\pi$: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes $K_n$. 

\[ \begin{array}{c}
\text{Diagram}
\end{array} \]
• Permutation graph = can be made from a permutation
  = comparability \cap co-comparability
  = comparability graphs of dimension 2 posets

• Lots of polynomial time algorithms here (e.g. MAXCLIQUE, TREEWIDTH)
Ordering permutations: containment

Example

- Pattern containment: a partial order, $\sigma \leq \pi$. 
Ordering permutations: containment

Example

- **Pattern containment**: a partial order, $\sigma \leq \pi$.
- **Draw the graphs**: $G_\sigma \leq_{\text{ind}} G_\pi$. 

![Graphs showing pattern containment]

1 3 5 2 4  

<  

4 2 1 6 3 8 5 7
Ordering permutations: containment

Example

- **Pattern containment**: a partial order, $\sigma \leq \pi$.
- Draw the graphs: $G_\sigma \leq_{\text{ind}} G_\pi$.
- **Permutation class**: downset in this ordering:
  
  $\pi \in C$ and $\sigma \leq \pi$ implies $\sigma \in C$.

- **Avoidance**: minimal forbidden permutation characterisation:
  
  $C = \text{Av}(B) = \{\pi : \beta \nleq \pi \text{ for all } \beta \in B\}$. 

WQO: Permutations $\leftrightarrow$ graphs

\[ \sigma \leq \pi \implies G_\sigma \leq_{\text{ind}} G_\pi \]

This means

\[
\text{Av}(B) \text{ is wqo} \implies \{G_\beta : \beta \in B\}\text{-free permutation graphs are wqo.}
\]

Conversely, the perm $\rightarrow$ graph mapping is not injective:

$P_4$ in two ways

\[
\text{Open Problem}
\]

\[
\text{Av}(B) \text{ is wqo} \iff \{G_\beta : \beta \in B\}\text{-free permutation graphs are wqo.}
\]
How to convert antichains

• For a graph $G$, define

$$\Pi(G) = \{\text{permutations } \pi : G_\pi \cong G\}.$$  

e.g. $\Pi(P_4) = \{2413, 3142\}$, and $\Pi(K_5) = \{54321\}$.

• Given a permutation antichain $A = \{\alpha_1, \alpha_2, \ldots\}$, want each $\Pi(G_{\alpha_i})$, to contain as few permutations as possible.

Fact

$G_{\alpha_i} \not\leq G_{\alpha_j}$ iff $\sigma \not\leq \alpha_j$ for all $\sigma \in \Pi(G_{\alpha_i})$.

• So for each $\sigma \in \Pi(G_{\alpha_i})$, it suffices to find $\tau \leq \sigma$ such that $\tau \not\leq \alpha_j$ for every $j$. 
Three permutation antichains required

- Size of forbidden path: $m$
- Size of forbidden clique: $n$
A $P_7, K_5$-free antichain

An antichain in $\text{Av}(54321, 2416375, 3152746)$ [Murphy, 2003]

For every $\pi$ in the above antichain:

• $|\Pi(G_\pi)| = 4$, and we know what they are.
• $\pi^{-1} \in \Pi(G_\pi)$ contains 51423, but $\pi$ does not.
• Other permutations in $\Pi(G_\pi)$ can be handled similarly.
The other two antichains

\[ P_6, K_6 \text{-free permutation graphs [B., 2012]} \]

\[ P_7, K_4 \text{-free permutation graphs [Murphy & Vatter, 2003]} \]
Wqo classes

- **Known:** $P_m, K_3$-free permutation graphs are wqo [Lozin and Mayhill, 2011]
**Wqo classes**

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- **Known:** $P_m, K_3$-free permutation graphs are wqo [Lozin and Mayhill, 2011]
- **Todo:** $P_5, K_n$-free permutation graphs are wqo, for all $n$. 
- $P_5, K_{126923785921975}$-free permutation graphs are wqo, but $P_5$-free permutation graphs are \textbf{not} wqo.

\begin{itemize}
  \item This antichain needs arbitrarily large cliques.
\end{itemize}
The permutation problem

Theorem

The class of permutations $Av(n \cdots 21, 24153, 31524)$ is wqo.

- $G_{n \cdots 21} \cong K_n$
- $G_{24153} \cong G_{31524} \cong P_5$ (and these are the only two permutations).
- So $Av(n \cdots 21, 24153, 31524)$ corresponds to $P_5, K_n$-free permutation graphs.

Corollary

The class of $P_5, K_n$-free permutation graphs is wqo.
Proposing the theorem – Step 1

Proposition

The simple permutations of $Av(n \cdots 21, 24153, 31524)$ are griddable.

- Simple permutations are ‘building blocks’ (c.f. prime graphs)
- Griddable = can draw on a picture like this:

![Diagram of griddable pattern]

Proof

- Induction on $n$.
- Key step: in graph terms, limit the size of the largest matching in a prime graph
What’s gridding good for?

**Theorem (Albert, Ruškuc, Vatter, 2014)**

If the *simple permutations* in a class are *geometrically griddable*, then the class is wqo.

‘Geometrically griddable’ is stricter than ‘griddable’

\[
\text{GGrid} \left( \begin{array}{ccc} & & \\ & | & \\ | & & \end{array} \right) \rightarrow P_4\text{-free split permutation graphs}
\]

is a subclass of:

\[
\text{Grid} \left( \begin{array}{ccc} & & \\ & | & \\ | & & \end{array} \right) \rightarrow \text{split permutation graphs}
\]

• Aim: take gridding from Step 1 and refine to a geometric one
Proposition

The *simple permutations* of $Av(n \cdots 21, 24153, 31524)$ are griddable without NW corners.

NW corners and cycles

- NW corner = configuration shown in red
Step 2 – refine the gridding

Proposition

The **simple permutations** of $Av(n \cdots 21, 24153, 31524)$ are griddable without **NW corners**.

NW corners and cycles

- NW corner = configuration shown in **red**
- Cycle = closed dotted line
- No NW corners $\Rightarrow$ no cycles!
- No cycles $\Rightarrow$ gridding is geometric $\Rightarrow$ class is wqo
• Three classes remain: \(\{P_6, K_5\}, \{P_6, K_4\} \text{ and } \{P_7, K_4\}\).
• Not griddable (in the sense used here)
• None of our antichain construction tricks work
Thanks!

Main reference:
Atminas, B., Korpelainen, Lozin & Vatter, Well-quasi-order for permutation graphs omitting a path and a clique, arXiv 1312:5907