Simplicity in Relational Structures

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Introduction

Outline

Sets and Relations

- A relational structure: a set of points, and a set of relations on these points.
- The ground set, A.
- A k-ary relation R – a subset of A^k.

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Permutations

- A permutation of length n is a structure on two linear relations.

Example

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
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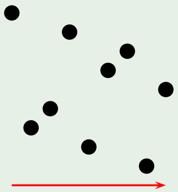

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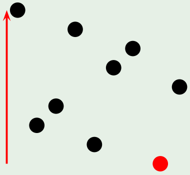

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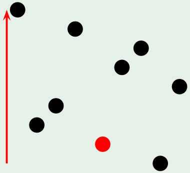

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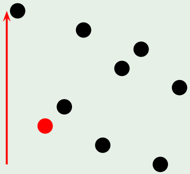

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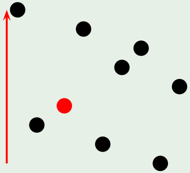

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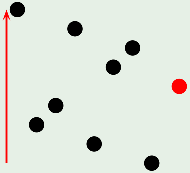

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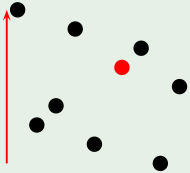

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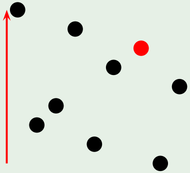

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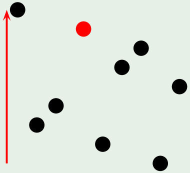

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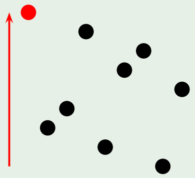

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Graphs

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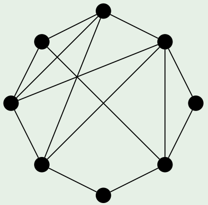

Posets

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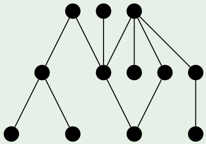

Tournaments

- A tournament is a complete oriented graph.
- As a relational structure, it is a single trichotomous binary relation. ($x \to y$, $y \to x$ or $x = y$.)
- A competition between players: $x \to y$ means "y wins."

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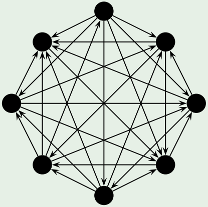

Intervals and Simplicity

- An interval: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, modules...
- A structure is simple if there are no proper intervals.
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Intervals

- Permutation π.
- An interval of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

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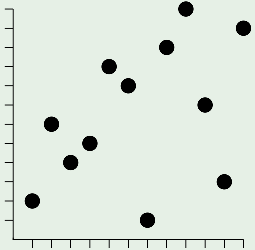

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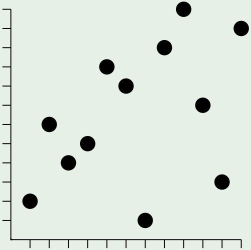

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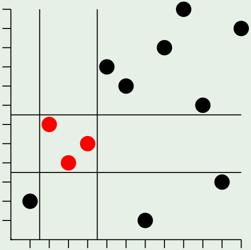

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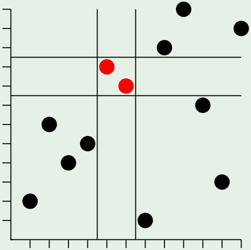

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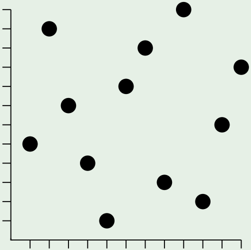

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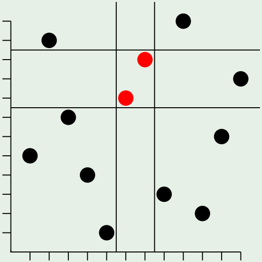

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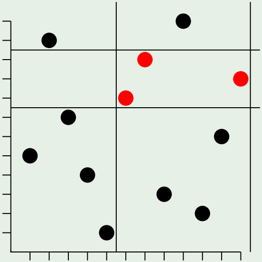

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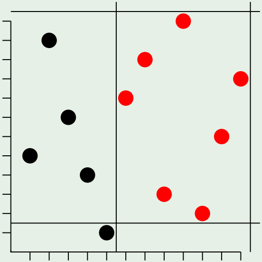

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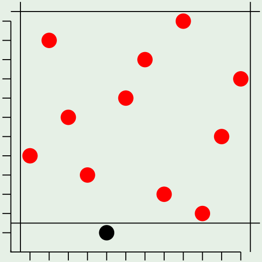

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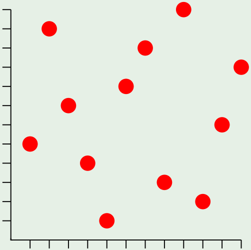

Simplicity in Graphs

- Simple graph?

Example

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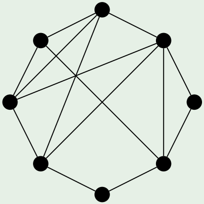

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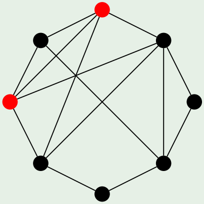

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Outline

Decomposing Permutations

- Every permutation has a block decomposition.
- Gives a unique simple permutation.

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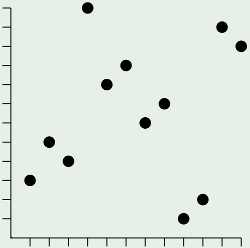

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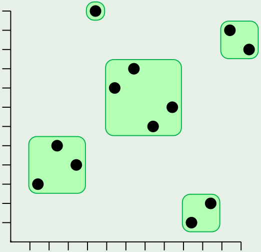

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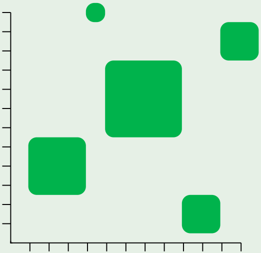

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- If simple has $>$ 2 points then the blocks are unique.
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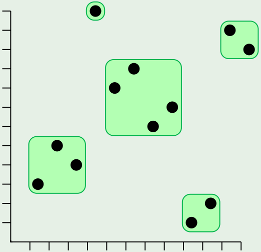

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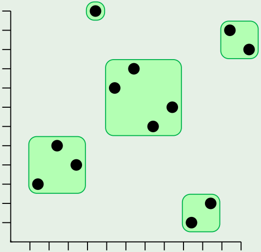

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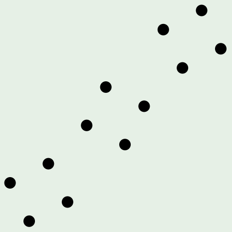

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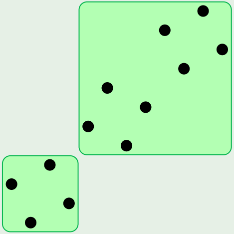

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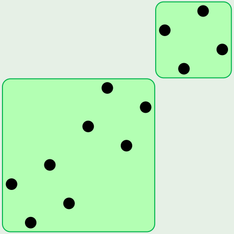

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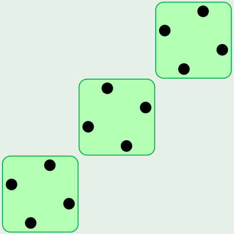

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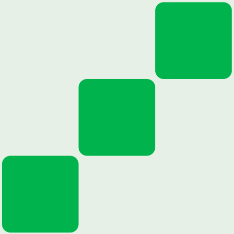

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- Increasing permutation: both linear orders agree.

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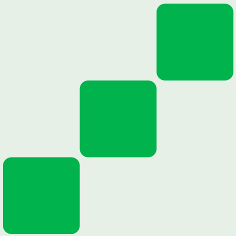

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- The substitution decomposition holds for every relational structure.
- Non-unique cases arise in two ways:
  - Degenerate – all relations complete or empty.
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- Graphs: degenerate only (complete or independent graph).
- Posets: linear (linear order).
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- More precisely:

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