Finite Basis Results of Wreath Products

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Introduction

Permutation Structure

Definition

For a permutation $\sigma = s_1 s_2 \ldots s_n$:

- A sequence is any set of symbols $s_{i_1}, s_{i_2}, \ldots, s_{i_k}$ from σ with $i_1 < i_2 < \ldots < i_k$.
- A segment is a sequence of adjacent symbols, $s_i, s_{i+1}, \ldots, s_{i+j}$.
- An interval or block of σ is a segment $s_i s_{i+1} \ldots s_{i+j}$, in which the set of values is contiguous:

- σ is irreducible if it contains no segment of the form $i, i+1$.

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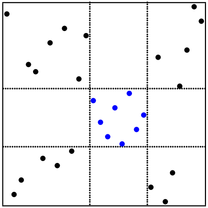

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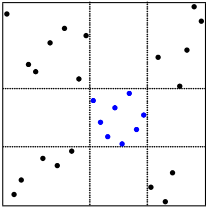

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Wreath Product I

Definition

The wreath product of the set of permutations X with the set of permutations Y is the set $X \wr Y$ of permutations

$$\sigma = \alpha_1 \alpha_2 \ldots \alpha_k$$

such that:

(i) each α_i is an interval,

(ii) each α_i is order isomorphic to a permutation of Y,

(iii) if for every i we pick a symbol a_i from α_i, then $a_1 a_2 \ldots a_k$ is order isomorphic to a permutation in X.

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Wreath Product II

Example

- $X = \{1, 12\}$, $Y = \{1, 12, 21\}$:

- $X \wr Y = \{1, 12, 21, 123, 132, 213, 1234, 1243, 2134, 2143\}$.

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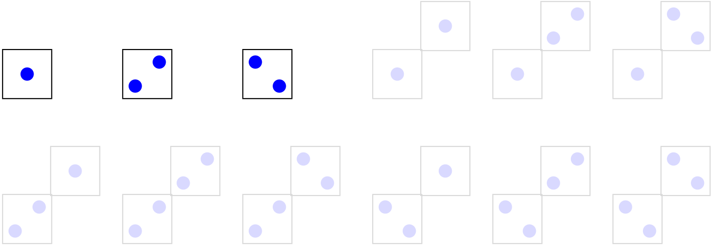

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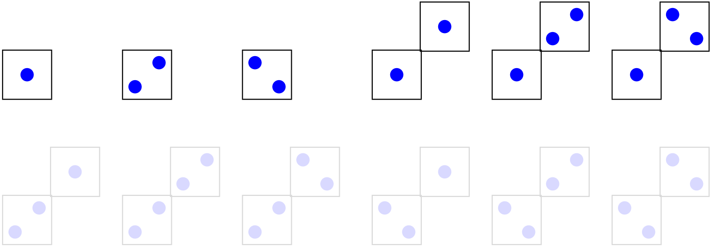

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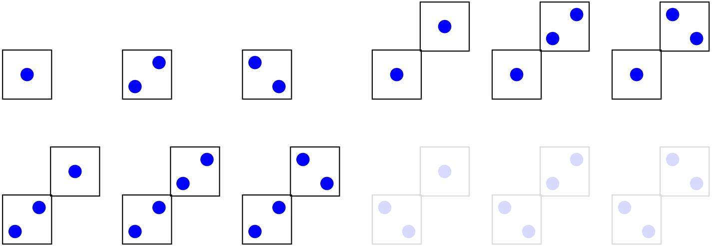

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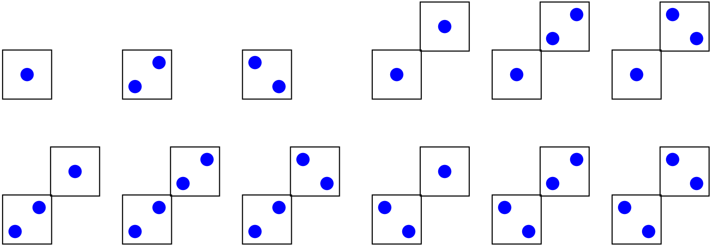

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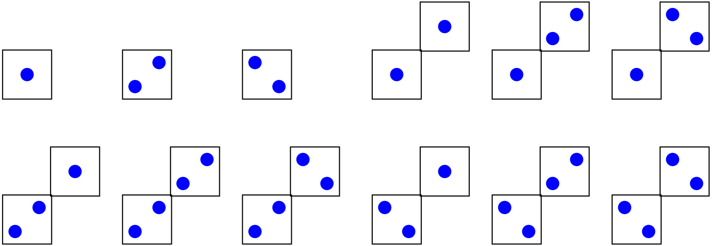

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Wreath Product III

- If X and Y are closed then $X \wr Y$ is closed.
- If X and Y are finitely based, is $X \wr Y$ finitely based?
- Not true – Atkinson proves $\mathcal{A}(21) \wr \mathcal{A}(321654)$ has infinite basis.
- Half the problem: which classes Y obey

$$X \text{ finitely based} \Rightarrow X \wr Y \text{ finitely based?}$$

Y has the Wreath Finite Basis Property (WFBP).

Example

$Y = \{1\}$. Then $X \wr Y = X$ for any class X.

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Profiles II

- View profile σ^* as “collapsing” maximal consecutive increasing sequences from σ.

Or ...

- “collapsing” maximal intervals of σ order isomorphic to elements from $I = \mathcal{A}(21) = \{1, 12, 123, \ldots\}$.

Generalised ...

- “collapsing” maximal intervals of σ order isomorphic to permutations from an arbitrary class Y.

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For any closed class Y, the permutation σ has Y-profile

$$\sigma^{(Y)} = s_1 s_2 \dots s_m$$

if σ can be partitioned into segments

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subject to

(i) each σ_i is a non-empty interval, order isomorphic to a permutation from Y,

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Made unique by first picking σ_1 maximally, then σ_2, then σ_3, etc.

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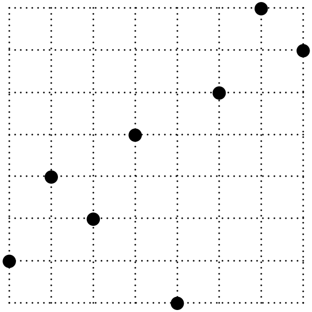

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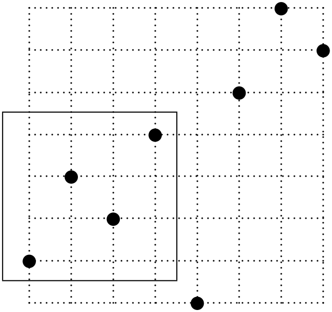

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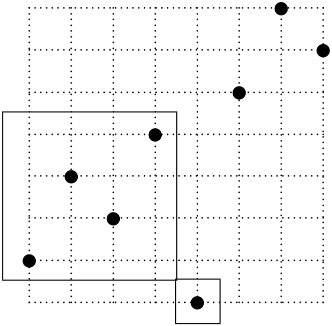

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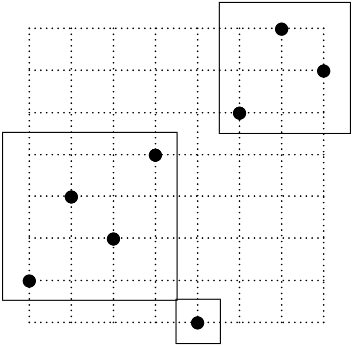

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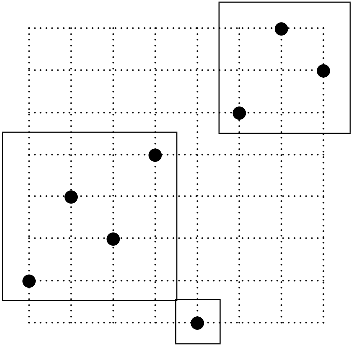

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Profiles & the Wreath Product

Theorem

For any closed classes X and Y,

$$\sigma \in X \wr Y \text{ if and only if } \sigma^{(Y)} \in X.$$

Proof

- Decompose σ into the intervals defined by the Y-profile, $\sigma = \sigma^{(Y)} \wr (\sigma_1, \sigma_2, \ldots, \sigma_k)$.
- Take any known decomposition $\sigma = \tau \wr (\tau_1, \tau_2, \ldots, \tau_l)$ with $\tau \in X$.
- "Superimpose" $\tau_1, \ldots, \tau_l$ over $\sigma_1, \ldots, \sigma_k$.
- Claim: Every σ_j has the right-hand end of some τ_{i_j} within it.
- Thus

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- Atkinson, Restricted Permutations and the Wreath Product, 2002.
- Lemma 1. β is a basis elements of $X \wr I$ if and only if β is minimal (under involvement) subject to
  (i) β is irreducible,
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- Lemma 2. Let β be any permutation and σ a permutation minimal subject to
  (i) $\beta \preccurlyeq \sigma$,
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  Then $|\sigma| \leq 2|\beta| - 1$.

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- By Lemma 2, we know $|\sigma'| \leq 2|\beta| - 1$. By Lemma 1, σ' must be a basis element of $X \wr I$.
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- How do we prove Lemma 2b?
  (i) Embed the permutation $\beta = b_1 \dots b_k$ as a subsequence $s_{i_1}, \dots, s_{i_k}$ of σ.
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- In general: When can we do this, and how many symbols need to be added?

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Definition

The minimal block of $\sigma = s_1 \ldots s_n$ containing symbols s_i and s_j (some i, j) is the smallest interval of σ containing both s_i and s_j.

- Denoted $\sigma^{\Diamond}_{i,j}$.
- $\sigma^{\Diamond}_{i,j}$ is unique for each pair (i, j).

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The left extension of σ with symbols s_i, s_j is the minimal position k such that $s_i < s_k < s_j$, or $s_j < s_k < s_i$, written $L_\sigma(i, j)$.

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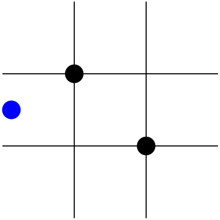

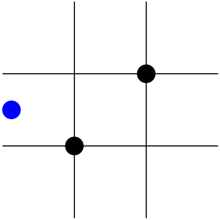

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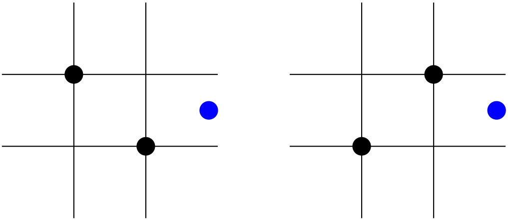

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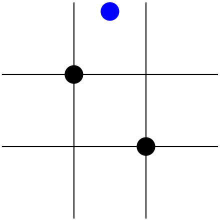

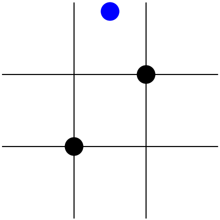

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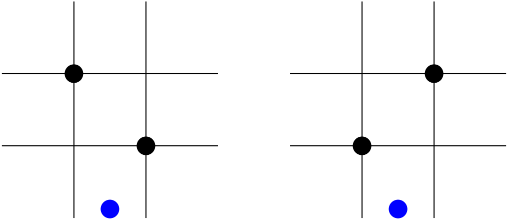

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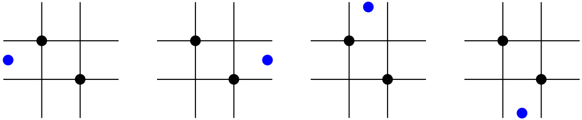

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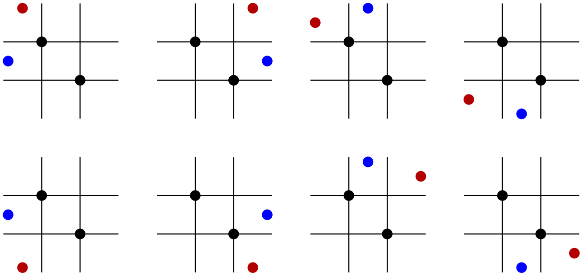

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- n-ary extensions may not exist. Must eventually reach the edges of the minimal block.

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An *n-ary extended block* of σ is the permutation formed by taking symbols with positions given by:

- An n-ary extension.
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- The primary "parent" extension, and the original i, j.

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The set of n-ary extended blocks of σ on pair (i, j) is $\mathcal{E}_\sigma(i, j; n)$. It is a subset of the generalised set of all 2^{n+2} possible n-ary extended blocks,

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- The primary "parent" extension, and the original i, j.

Definition

The set of n-ary extended blocks of σ on pair (i, j) is $\mathcal{E}_\sigma(i, j; n)$. It is a subset of the generalised set of all 2^{n+2} possible n-ary extended blocks,

$$\mathcal{E}(n).$$

WFBP & Extended Blocks I

Theorem

Let $Y = \mathcal{A}(\beta_1, \ldots, \beta_m)$, and suppose

$$\exists\, q \text{ s.t. } \forall\, \varepsilon \in \mathcal{E}(q), \exists\, k \in \{1, \ldots, m\} \text{ s.t. } \beta_k \preccurlyeq \varepsilon.$$

Then Y possesses the WFBP.

Proof.

- Invoke Lemma 2b: embed basis elements of a class X within basis elements of $X \wr Y$.
- For each pair $s_{i_j}, s_{i_{j+1}}$, the minimal block $\sigma^{\Diamond}_{i_j, i_{j+1}}$ must involve a basis element β_k of Y.
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WFBP & Extended Blocks II

Proof (ctd).

Or:

(ii) No q-ary extensions exist: we reach the boundaries of $\sigma^{\Diamond}_{i_j,i_{j+1}}$. Then β_k appears within these boundaries, and separates s_{i_j} from $s_{i_{j+1}}$.

- Thus we bound basis elements ω of $X \wr Y$ by

$$|\omega| \leq p + (2(q-1) + r)(p-1)$$

where:

(i) p = maximum length of basis elements in X.
(ii) r = maximum length of basis elements in Y.

- Hence $X \wr Y$ is finitely based.

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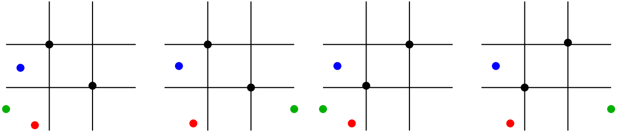
Applications: Classes with WFBP

- Stack sortable permutations, $\mathcal{A}(231)$. Every $\varepsilon \in \mathcal{E}(3)$ involves 231.



- Separable permutations, $\mathcal{A}(2413, 3142)$. Every $\varepsilon \in \mathcal{E}(3)$ involves 2413 or 3142.
- $\mathcal{A}(\beta)$ for $\beta \in \{132, 312, 213\}$.
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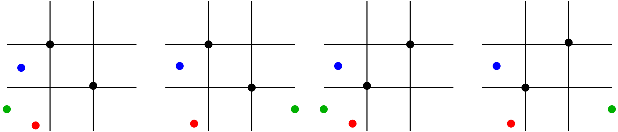

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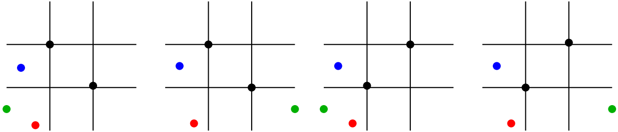

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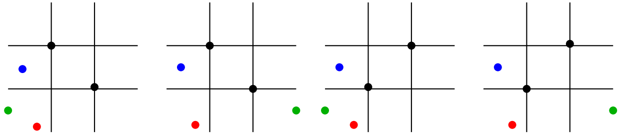

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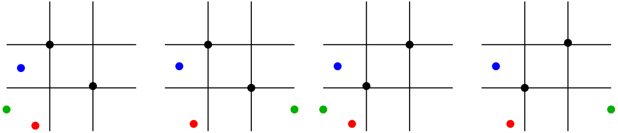

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- Extensions sufficient, but necessary?
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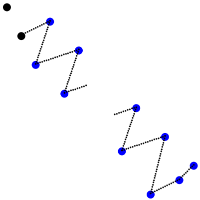

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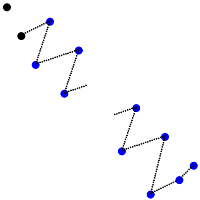

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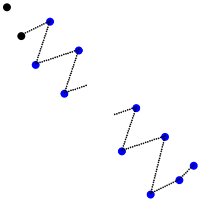

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