Grid Class Enumeration Techniques

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Friday 24th June, 2011

(Joint with Michael Albert and Mike Atkinson)

Counting…

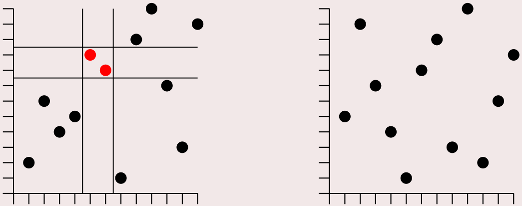
- $\text{Av}(B)$ = class of permutations avoiding the set of permutations B (in the Graeco-Roman sense).
- What is $\displaystyle\sum_{\pi \in \text{Av}(B)} x^{|\pi|}$?
- Many techniques (some we have seen this week). Here's another...

. . . as easy as 1 2 3

1. Count the simple permutations.
2. Work out how to inflate the simples.
3. Substitute the inflations into the simples, and finish off.

Simple Permutations

- Interval: maps contiguous positions to contiguous values.
- Simple permutation: only intervals are singletons and the whole thing.



Not simple Simple

- Enumerate classes by dividing them up:

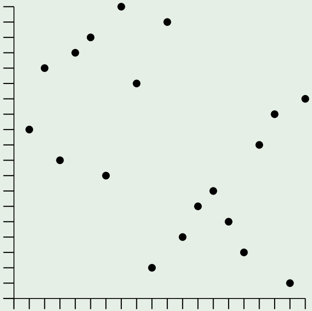
$$\mathrm{Av}(B) = \{1\} \cup \{\oplus\text{–decomposables}\} \cup$$
$$\cup \{\ominus\text{–decomposables}\} \cup \{\text{inflations of simples}\}$$

(Monotone) Grid Classes

- Matrix $\mathcal{M}$ whose entries are permutation classes.
- Today: all non-$\emptyset$ cells are $\mathrm{Av}(21)$ or $\mathrm{Av}(12)$.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



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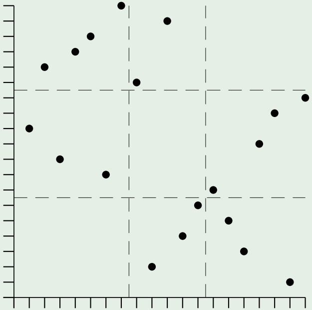
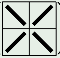


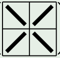
Geometric Grid Classes

- Fill a square grid with 45° lines.
- Make permutations by choosing points from these lines.
- These are not just monotone grid classes:

Example

Geom() $= \mathrm{Av}(2143, 2413, 3142, 3412)$

is a subclass of:

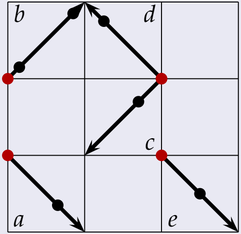
Grid() $= \mathrm{Av}(2143, 3412)$

Geometric enumeration

Theorem (Albert, Atkinson, Bouvel, Ruškuc and Vatter)

Geometric grid classes can be encoded by a regular language, and therefore have rational generating functions.

Proof.



$4\,1\,6\,5\,3\,2 \mapsto$

(Homage to Nik Ruškuc for the illustration.) □

Practical enumeration

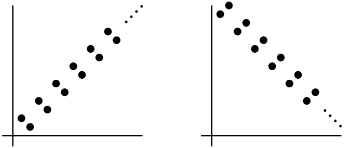
- Test ground: count classes avoiding two permutations of length 4.
- Up to symmetry, four such classes remain that are monotone griddable:

$$\mathrm{Av}(1324, 4312) \qquad \mathrm{Av}(2143, 4231)$$
$$\mathrm{Av}(2143, 4312) \qquad \mathrm{Av}(2143, 4321)$$

- Each class is the union of several geometric grid classes.

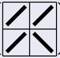
Theorem [Huczynska & Vatter, 2006]

A class $\mathcal{C}$ is monotone griddable if and only if it contains neither the classes $\oplus 21$ nor $\ominus 12$.

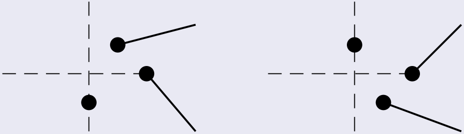


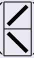
Enumerating Av(2143, 4312)

Lemma

Av(2143, 4312) *is contained in Grid* ().

Proof.



- If $\pi \in$ Grid () $=$ Av(132, 312) then done.
- Scan $\pi \in$ Av(2143, 4312) from right to left. Stop at first 132 or 312.

□

Enumerating Av(2143, 4312)

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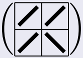
$Av(2143, 4312)$ is contained in Grid.

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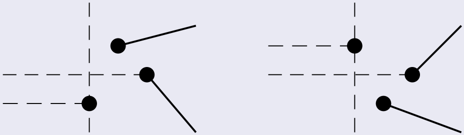
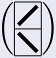
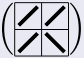


- If $\pi \in$ Grid $=$ Av$(132, 312)$ then done.
- In either case, three regions on left hand side.

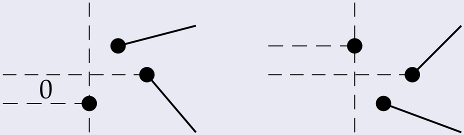
□

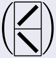
Enumerating Av(2143, 4312)

Lemma

Av(2143, 4312) *is contained in Grid* .

Proof.



- If $\pi \in$ Grid  $=$ Av(132, 312) then done.
- 132: Regions are monotone or empty to avoid 2143, 4312.

□

Enumerating Av(2143, 4312)

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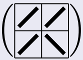
Av(2143, 4312) *is contained in Grid* .

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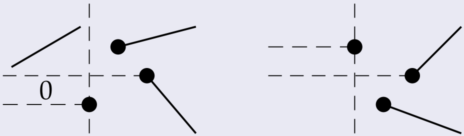
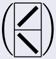


- If $\pi \in$ Grid  = Av(132, 312) then done.
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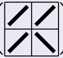
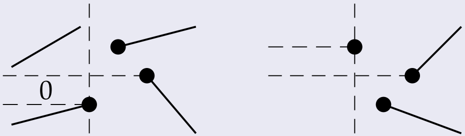
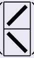
Av(2143, 4312) *is contained in Grid*().

Proof.



- If $\pi \in$ Grid() = Av(132, 312) then done.
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Enumerating Av(2143, 4312)

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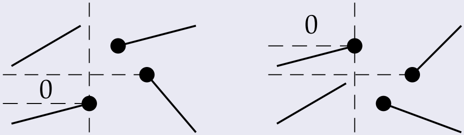
Av(2143, 4312) *is contained in Grid* $\left(\begin{array}{|c|c|}\hline \diagup & \diagup \\ \hline \diagup & \diagdown \\ \hline \end{array}\right)$.

Proof.

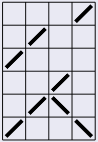
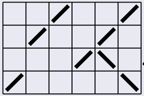


- If $\pi \in \mathrm{Grid}\left(\begin{array}{|c|}\hline \diagup \\ \hline \diagdown \\ \hline \end{array}\right) = \mathrm{Av}(132, 312)$ then done.
- 312: Similar.

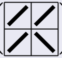
□

Lemma

Av(2143, 4312) *is equal to*

 $\cup$ .

Proof.

- 4312 is a basis element of Grid ().
- Look at embeddings of 2143 — what does this exclude?

□

Finishing off Av(2143, 4312)

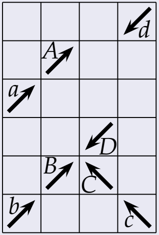
Theorem (Albert, Atkinson, B., 2011)

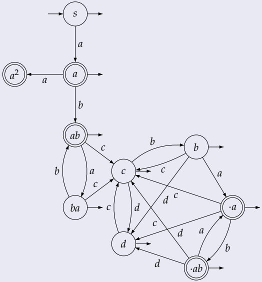
Av(2143, 4312) *has generating function*

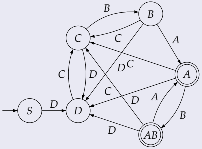
$$\frac{1 - 13x + 69x^2 - 191x^3 + 294x^4 - 252x^5 + 116x^6 - 23x^7}{(1-x)^2(1-3x)^2(1-3x+x^2)^2}$$

Proof idea

Encode the simples:







Another approach...

Previously: Av$(2143, 4312)$

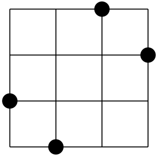
- Found a quick argument to describe the structure of the class.
- Used the class structure to enumerate the simples.
- Play with generating functions.

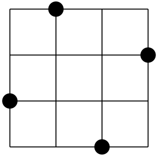
Next: Av$(2143, 4231)$

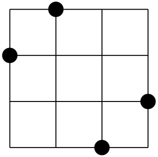
- Describe the structure of the simples, and enumerate them.
- Establish how simples can be inflated.
  (Corollary: structure of the class.)
- Brief play with generating functions.

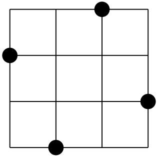
Av(2143, 4231) – the simples

Every simple permutation has four distinct extremal points, in one of four configurations:



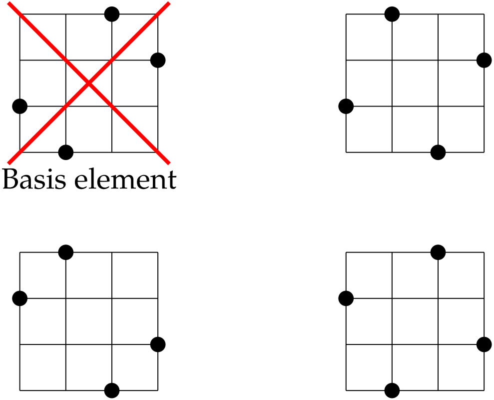






Av(2143, 4231) – the simples

Every simple permutation has four distinct extremal points, in one of four configurations:



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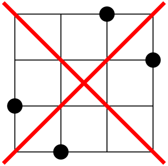
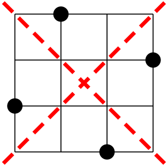
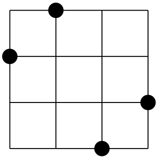
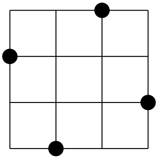


Basis element



Symmetry of below





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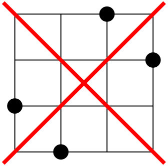

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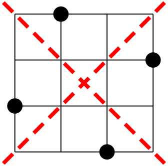

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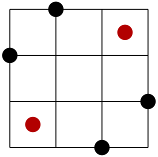
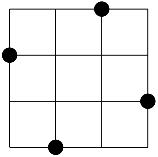


$= 35142$ or 42513



Av(2143, 4231) – the simples

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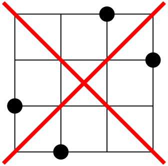

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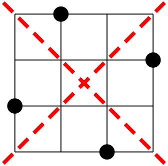
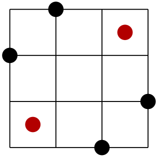


Symmetry of below



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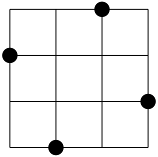
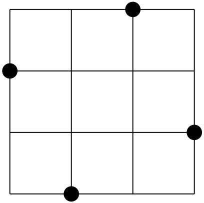


Diagram chase…

Chasing diagrams in Av(2143, 4231)

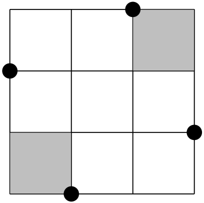
Simple permutation whose extremal points are 3142:



- Study basis and simplicity conditions on cells.

Chasing diagrams in Av(2143, 4231)

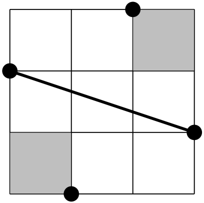
Simple permutation whose extremal points are 3142:



- Two cells empty to avoid 2143.

Chasing diagrams in Av(2143, 4231)

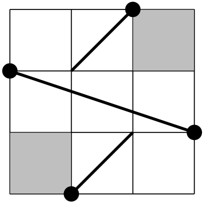
Simple permutation whose extremal points are 3142:



- Decreasing to avoid 4231.

Chasing diagrams in Av(2143, 4231)

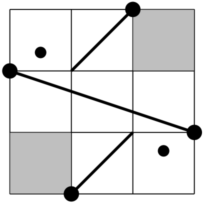
Simple permutation whose extremal points are 3142:



- Two increasing cells, to avoid 2143.

Chasing diagrams in Av(2143, 4231)

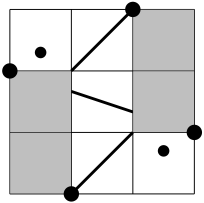
Simple permutation whose extremal points are 3142:



- Non obvious: At most one point in TL and BR.

Chasing diagrams in Av(2143, 4231)

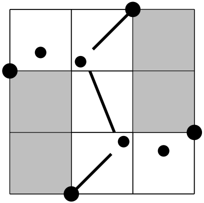
Simple permutation whose extremal points are 3142:



- Non obvious: Two empty cells.

Chasing diagrams in Av(2143, 4231)

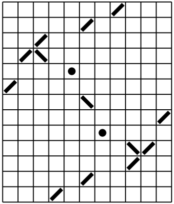
Simple permutation whose extremal points are 3142:

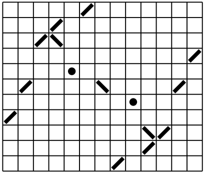


- Avoiding 4231 leaves us with this.

Enumerating Av(2143, 4231)

Studying inflations, this class is the union of:





Enumeration is now easy:

Theorem (Albert, Atkinson, B., 2010)

$Av(2143, 4231)$ has generating function

$$\frac{1 - 12x + 60x^2 - 162x^3 + 259x^4 - 252x^5 + 146x^6 - 46x^7 + 8x^8}{(1-x)^4(1-3x)(1-3x+x^2)^2}$$

Two more classes, and closing remarks

- Av(2143, 4321): Structure is established (A.& V.), but haven't bothered to do the enumeration (yet).
- Av(1324, 4312): Proving trickier…
- Future aim: To turn these ad hoc "diagram chases" into something routine/automatic.

Thanks!