Forbidding paths and cliques
graphs, permutation graphs, and well-quasi-ordering

Robert Brignall
Joint work with Atminas, Korpelainen, Lozin and Vatter

2nd July 2013
Orderings on Structures

- Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, …
Orderings on Structures

- Pick your favourite **family of combinatorial structures**. E.g. graphs, permutations, tournaments, posets, ...
- Give your family an **ordering**. E.g. graph minor, induced subgraph, permutation containment, ...
Orderings on Structures

• Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, …
• Give your family an ordering. E.g. graph minor, induced subgraph, permutation containment, …
• Does your ordering contain infinite antichains? i.e. an infinite set of pairwise incomparable elements.

Example ((Induced) subgraph antichains)
Cycles:

“Split end” graphs:
When are there antichains?

No infinite antichains = well-quasi-ordered.

- **Words** over a finite alphabet with subword ordering [Higman, 1952].
- **Trees** ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- **Graphs closed under minors** [Robertson and Seymour, 1983—2004].

Infinite antichains.

- **Graphs closed under induced subgraphs** (or merely subgraphs).
- Permutations closed under **containment**.
- Tournaments, digraphs, posets, . . . with their natural **induced substructure** ordering.
Downsets

**Question**

*In your favourite ordering, which downsets contain infinite antichains?*

- Downset (or *hereditary property*): set \( \mathcal{P} \) of objects such that
  
  \[ G \in \mathcal{P} \text{ and } H \leq G \text{ implies } H \in \mathcal{P}. \]

  e.g. triangle-free graphs — (induced) subgraph ordering.

- **Today**: (permutation) graphs with no large clique or long path as an induced subgraph.
Forbidding paths and cliques

- = Graphs wqo
- = Permutation graphs wqo, graphs not wqo
- = Permutation graphs not wqo
Permutation graphs

- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph $G_\pi$: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes $K_n$. 
Permutation graphs

• Permutation $\pi = \pi(1) \cdots \pi(n)$
• Make a graph $G_{\pi}$: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$.
• Note: $n \cdots 21$ becomes $K_n$. 
Permutation graphs

- Permutation graph = can be made from a permutation.
  = comparability ∩ co-comparability
  = comparability graphs of dimension 2 posets

- Lots of polynomial time algorithms here (largest clique, tree width, clique width)
Lemma

P₇, K₅-free permutation graphs contain an infinite antichain.

Proof.

Here's an element of an infinite permutation antichain:

Turn this into a graph, show it is:

1. an antichain
2. P₇, K₅-free.
Where are we?

• Done: $P_7, K_n$-free permutation graphs not wqo ($n \geq 5$)
Where are we?

- **Done:** $P_7$, $K_n$-free permutation graphs not wqo ($n \geq 5$)
- **Next:** $P_5$, $K_n$-free permutation graphs are wqo, for all $n$. 
• $P_5$, $K_{126923785921975}$-free permutation graphs are wqo, but $P_5$-free permutation graphs are not wqo.

Here’s an antichain element

• $P_5$, $K_4$-free graphs are not wqo [Korpelainen and Lozin]
To permutations...

Theorem

The class of permutations $Av(n \cdots 21, 24153, 31524)$ is wqo.

- $G_{n \cdots 21} \cong K_n$
- $G_{24153} \cong G_{31524} \cong P_5$ (and these are the only two permutations).
- So $Av(n \cdots 21, 24153, 31524)$ corresponds to $P_5, K_n$-free permutation graphs.
- $\sigma \leq \pi$ implies $G_\sigma \leq G_\pi$, so:

Corollary

The class of $P_5, K_n$-free permutations graphs is wqo.
And how to prove the theorem?

Two steps:

**Step 1 (Induction on \( n \))**

The **simple permutations** of \( \text{Av}(n \cdots 21, 24153, 31524) \) are griddable.

- Simple permutations = ‘building blocks’ of the class
- Griddable = can plot the permutations on a picture like this:

```
/  \
/   \
/     \
/       \
/         \
/           \
/             \
/               \
```

(For more on griddings, see David Bevan’s talk on Thursday.)
And how to prove the theorem?

Two steps:

Step 2 (Refine the gridding)

The **simple permutations** of $\text{Av}(n \cdots 21, 24153, 31524)$ are griddable without NW corners.

- NW corner = cells in this configuration:
What use is that?

• Murphy and Vatter, 2003:

Cycles are BAD

\[ \begin{array}{ccc}
\hline
\hline
\hline
\hline
\hline
\end{array} \]

= Infinite antichains!
What use is that?

• Murphy and Vatter, 2003:

Cycles are BAD

= Infinite antichains!

• Cycles have NW corners, so we have no cycles!

(The simples of) $\text{Av}(n \cdots 21, 24153, 31524)$ are good

• No cycles = wqo
• Simples good $\Rightarrow$ whole class is wqo [Albert, Ruškuc, Vatter]
Done: $P_7$, $K_n$-free permutation graphs not wqo ($n \geq 5$)
Done: $P_5$, $K_n$-free permutation graphs are wqo, for all $n$. 
Where are we?

- **Done**: $P_7$, $K_n$-free permutation graphs not wqo ($n \geq 5$)
- **Done**: $P_5$, $K_n$-free permutation graphs are wqo, for all $n$.
- **Lastly**: The gap?
$P_6, K_n$-free permutation graphs

- Not griddable (in the sense defined here)
- None of our antichain construction tricks work

**Tentative Conjecture**

$P_6, K_n$-free permutation graphs are wqo.
“The fewer questions you ask, the sooner we get wine.”

— Vincent Vatter, Permutation Patterns 2013, speaking about now