Forbidding paths and cliques

graphs, permutation graphs, and well-quasi-ordering

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Joint work with Atminas, Korpelainen, Lozin and Vatter

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Orderings on Structures

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  E.g. graphs, permutations, tournaments, posets, ...

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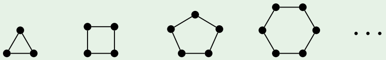
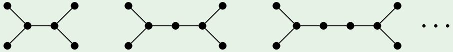
- Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, ...
- Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment, ...
- Does your ordering contain infinite antichains?
  i.e. an infinite set of pairwise incomparable elements.

Example ((Induced) subgraph antichains)

Cycles:



"Split end" graphs:



When are there antichains?

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No infinite antichains = well-quasi-ordered.

- Words over a finite alphabet with subword ordering [Higman, 1952].
- Trees ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- Graphs closed under minors [Robertson and Seymour, 1983—2004].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs).
- Permutations closed under containment.
- Tournaments, digraphs, posets, . . . with their natural induced substructure ordering.

Downsets

Question

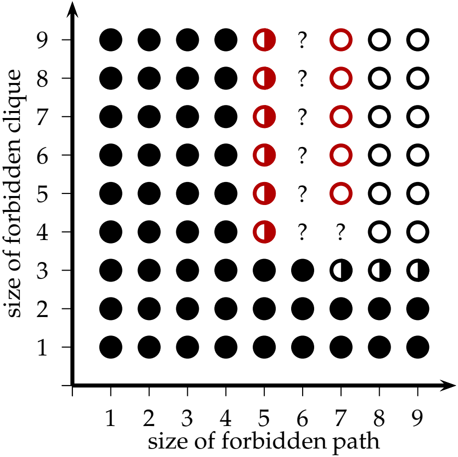
In your favourite ordering, which downsets contain infinite antichains?

- Downset (or hereditary property): set $\mathcal{P}$ of objects such that

$$G \in \mathcal{P} \text{ and } H \leq G \text{ implies } H \in \mathcal{P}.$$

  e.g. triangle-free graphs — (induced) subgraph ordering.
- Today: (permutation) graphs with no large clique or long path as an induced subgraph.

Forbidding paths and cliques

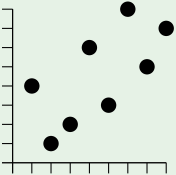


● = Graphs wqo

◑ = Permutation graphs wqo, graphs not wqo

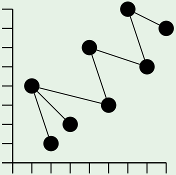
○ = Permutation graphs not wqo

Permutation graphs



- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph G_π: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes K_n.

Permutation graphs



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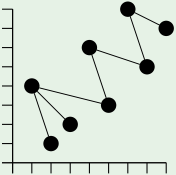


- Permutation graph = can be made from a permutation.
  = comparability $\cap$ co-comparibility
  = comparability graphs of dimension 2 posets
- Lots of polynomial time algorithms here (largest clique, tree width, clique width)

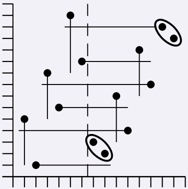
The easy bit

Lemma

P_7, K_5-free permutation graphs contain an infinite antichain.

Proof.

Here's an element of an infinite permutation antichain:

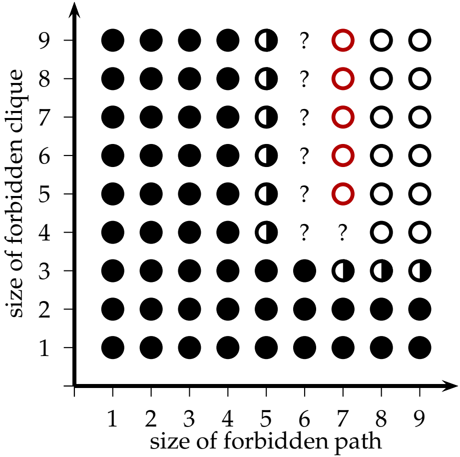


Turn this into a graph, show it is:

1. an antichain
2. P_7, K_5-free.

□

Where are we?



- Done: P_7, K_n-free permutation graphs not wqo ($n \geq 5$)

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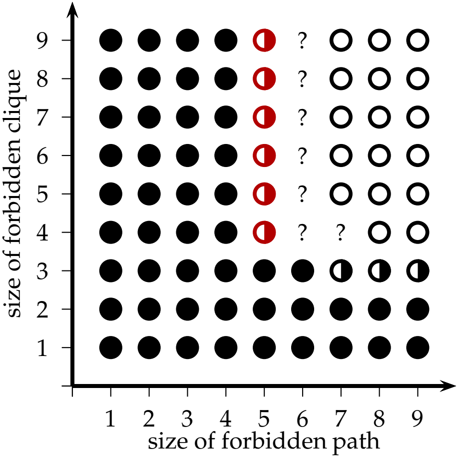
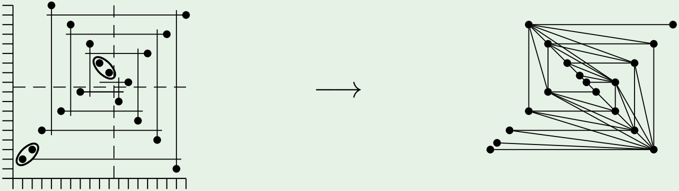


- Done: P_7, K_n-free permutation graphs not wqo ($n \geq 5$)
- Next: P_5, K_n-free permutation graphs *are* wqo, for all n.

Notes

- $P_5, K_{126923785921975}$-free permutation graphs are wqo, but P_5-free permutation graphs are not wqo.

Here's an antichain element



- P_5, K_4-free graphs are not wqo [Korpelainen and Lozin]

To permutations. . .

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Theorem

The class of permutations $Av(n \cdots 21, 24153, 31524)$ *is wqo.*

- $G_{n \cdots 21} \cong K_n$
- $G_{24153} \cong G_{31524} \cong P_5$ (and these are the only two permutations).
- So $\text{Av}(n \cdots 21, 24153, 31524)$ corresponds to P_5, K_n-free permutation graphs.
- $\sigma \leq \pi$ implies $G_\sigma \leq G_\pi$, so:

Corollary

The class of P_5*,* K_n*-free permutations graphs is wqo.*

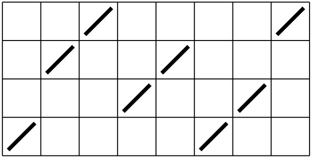
And how to prove the theorem?

Two steps:

Step 1 (Induction on n)

The simple permutations of $\mathrm{Av}(n \cdots 21, 24153, 31524)$ are griddable.

- Simple permutations = 'building blocks' of the class
- Griddable = can plot the permutations on a picture like this:



(For more on griddings, see David Bevan's talk on Thursday.)

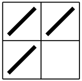
And how to prove the theorem?

Two steps:

Step 2 (Refine the gridding)

The simple permutations of $\mathrm{Av}(n \cdots 21, 24153, 31524)$ are griddable without NW corners.

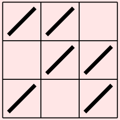
- NW corner = cells in this configuration:



What use is that?

- Murphy and Vatter, 2003:

Cycles are BAD

 = Infinite antichains!

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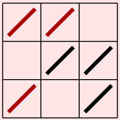
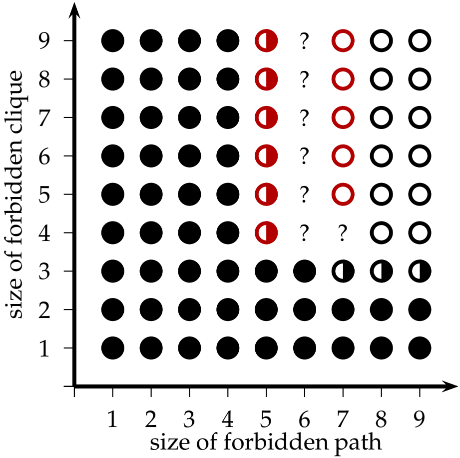
 = Infinite antichains!

- Cycles have NW corners, so we have no cycles!

(The simples of) $\mathrm{Av}(n \cdots 21, 24153, 31524)$ are good

- No cycles = wqo
- Simples good $\Rightarrow$ whole class is wqo [Albert, Ruškuc, Vatter]

Where are we?



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- Done: P_5, K_n-free permutation graphs *are* wqo, for all n.

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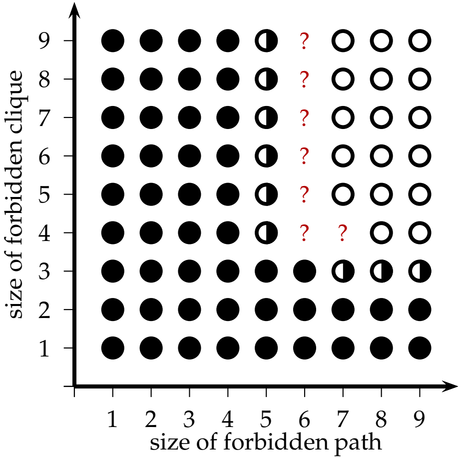

- Done: P_7, K_n-free permutation graphs not wqo ($n \geq 5$)
- Done: P_5, K_n-free permutation graphs *are* wqo, for all n.
- Lastly: The gap?

P_6, K_n-free permutation graphs

- Not griddable (in the sense defined here)
- None of our antichain construction tricks work

Tentative Conjecture

P_6, K_n-free permutation graphs are wqo.

"The fewer questions you ask, the sooner we get wine."

— Vincent Vatter, Permutation Patterns 2013, speaking about now