From permutations to graphs
well-quasi-ordering and infinite antichains

Robert Brignall
Joint work with Atminas, Korpelainen, Lozin and Vatter

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Orderings on Structures

• Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, …
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• Give your family an ordering. E.g. graph minor, induced subgraph, permutation containment, …
Orderings on Structures

• Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, ...

• Give your family an ordering. E.g. graph minor, induced subgraph, permutation containment, ...

• Does your ordering contain infinite antichains? i.e. an infinite set of pairwise incomparable elements.

Example ((Induced) subgraph antichains)

Cycles:

```
    A  B  C  D  ...
```

“Split end” graphs:

```
      A -- B -- C
      |    |    |
      X    X    X
      |    |    |
      Y    Y    Y
```


When are there antichains?

No infinite antichains = well-quasi-ordered.

• Words over a finite alphabet with subword ordering [Higman, 1952].
• Trees ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
• Graphs closed under minors [Robertson and Seymour, 1983—2004].

Infinite antichains.

• Graphs closed under induced subgraphs (or merely subgraphs).
• Permutations closed under containment.
• Tournaments, digraphs, posets, . . . with their natural induced substructure ordering.
Why study this?

For grant-writing
Algorithms inside well-quasi-ordered sets
- Polynomial-time recognition: is one graph a minor of another?
- Fixed-parameter tractability: e.g. graphs with vertex cover at most \( k \) can be recognised in polynomial time.

For mathematicians
- Well-quasi-order = nice structure. Useful for other problems (e.g. enumeration)
- Connections with logic: Kruskal’s Tree Theorem is unprovable in Peano arithmetic [Friedman, 2002]
- Antichains are pretty! (See later)
- It is fun [Kříž and Thomas, 1990]
- *Because it’s there.* [Mallory]
Formal definition

- Quasi order: reflexive transitive relation.
- Partial order: quasi order + asymmetric.

**Definition**

Let \((S, \leq)\) be a quasi-ordered (or partially-ordered) set. Then \(S\) is said to be well quasi ordered (wqo) under \(\leq\) if it

- is **well-founded** (no infinite descending chain), and
- contains no infinite antichain (set of pairwise incomparable elements).

Well founded: usually trivial for finite combinatorial objects, so this is all about the antichains.
Don’t panic! Maybe you could restrict to a subcollection?

Example: Cographs as induced subgraphs

Cographs = graphs containing no induced $P_4$
= closure of $K_1$ under complementation and disjoint union.

Cographs are well-quasi-ordered. [Damaschke, 1990]

Learn to stop worrying and love the antichains! [sorry, Kubrick]
Downsets

Question

In your favourite ordering, which downsets contain infinite antichains?

• Downset (or hereditary property, or class): set $C$ of objects such that $G \in C$ and $H \leq G$ implies $H \in C$.

Examples

• Triangle-free graphs: downset under (induced) subgraphs. Not wqo.
• Cographs: downset under induced subgraphs. Wqo.
• Planar graphs: downset under graph minor. Wqo.
• Words over \{0, 1\} with no ‘00’ factor: downset under factor order. Not wqo: 010, 0110, 01110, 011110, . . .
Minimal forbidden elements

- Downsets often defined by the **minimal forbidden elements**.

**Examples**

- Triangle-free graphs: $K_3$ free as (induced) subgraph.
- Cographs: Free($P_4$).
- Planar graphs: $\{K_5, K_{3,3}\}$-minor free graphs [Wagner’s Theorem]
- Pattern-avoiding permutations: Av($321$) (see later).

- Confusingly, the set of minimal forbidden elements is an antichain!
- Graph Minor Theorem $\Rightarrow$ every minor-closed class has finitely many forbidden elements.
Decision procedures

Question

In your favourite ordering, which downsets contain infinite antichains?

Known decision procedures

• **Graph minors**: no antichains anywhere!
• **Subgraph order**: a downset is wqo if and only if it contains neither \( \triangle, \Box, \bigcirc \ldots \) nor \( \prec \prec \prec \ldots \) [Ding, 1992]
• **Factor order**: downsets of words over a finite alphabet [Atminas, Lozin & Moshkov, 2013]

Theorem (Cherlin & Latka, 2000)

Any downset with \( k \) minimal forbidden elements is wqo iff it doesn’t contain any of the infinite antichains in a finite collection \( \Lambda_k \).
Plan for the rest of today

Question

For which $m, n$ is the following true?

The set of permutation graphs with no induced $P_m$ or $K_n$ is wqo.

We’ll:

- Build some antichains;
- Find structure to prove wqo.

Motivation?

- The ‘right’ level of difficulty: Interestingly complex, but tractable.
- Demonstration of some recently-developed structural theory.
- Expansion of the graph $\leftrightarrow$ permutation interplay.
Forbidding paths and cliques

\[
\begin{array}{cccccccccc}
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & \\
\end{array}
\]

- ● = Graphs wqo
- ○ = Permutation graphs wqo, graphs not wqo
- ○ = Permutation graphs not wqo

Graphs wqo
Permutation graphs wqo, graphs not wqo
Permutation graphs not wqo
These classes are trivially wqo.
Permutation graphs

Cographs are wqo [Damaschke, 1990]
Permutation graphs

$P_6, K_3$-free graphs are wqo [Atminas and Lozin, 2014+]

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{permutation_graphs}
\caption{Permutation graph representation with forbidden path and clique sizes.}
\end{figure}
Permutation graphs

$P_5, K_4$-free graphs are not wqo [Korpelainen and Lozin, 2011]
Permutation graphs

$P_7, K_3$-free graphs are not wqo [Korpelainen and Lozin, 2011b]
Permutation graphs

- Permutation $\pi = \pi(1) \cdots \pi(n)$
- Make a graph $G_\pi$: for $i < j$, $ij \in E(G_\pi)$ iff $\pi(i) > \pi(j)$.
- Note: $n \cdots 21$ becomes $K_n$. 
Permutation graphs

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Permutation graphs

- Permutation graph can be made from a permutation
  \[ \text{comparability} \cap \text{co-comparibility} \]
  \[ \text{comparability graphs of dimension 2 posets} \]

- Lots of polynomial time algorithms here (largest clique, tree width, clique width)
Forbidding paths and cliques

- $m$: size of forbidden path
- $n$: size of forbidden clique

- $=\text{Graphs wqo}$
- $\circ=\text{Permutation graphs wqo, graphs not wqo}$
- $\circ=\text{Permutation graphs not wqo}$
Ordering permutations: containment

Example

- Pattern containment: a partial order, $\sigma \leq \pi$. 
Example

- **Pattern containment**: a partial order, $\sigma \leq \pi$.
- **Draw the graphs**: $G_{\sigma} \leq_{\text{ind}} G_{\pi}$.
Ordering permutations: containment

Example

- **Pattern containment**: a partial order, $\sigma \leq \pi$.
- **Draw the graphs**: $G_\sigma \leq_{\text{ind}} G_\pi$.
- **Permutation class**: downset in this ordering:

  $$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$ 

- **Avoidance**: minimal forbidden permutation characterisation:

  $$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$
This means

\[ \text{Av}(B) \text{ is wqo } \iff \{ G_\beta : \beta \in B \} -\text{free permutation graphs are wqo.} \]

Conversely, the perm → graph mapping is not injective:

\[ P_4 \text{ in two ways} \]

\[ \text{Av}(B) \text{ is wqo } \iff \{ G_\beta : \beta \in B \} -\text{free (permutation) graphs are wqo.} \]
Antichains: permutations $\leftrightarrow$ graphs

Open Problem

$\text{Av}(B)$ is wqo $\iff \{G_\beta : \beta \in B\}$-free (permutation) graphs are wqo.

- Despite this, lots of permutation antichains seem to translate...

Increasing oscillations $=$ split end graphs

- ... and there are lots of permutation antichains to choose from!
Antichains: permutations $\leftrightarrow$ graphs

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Increasing oscillations = split end graphs

... and there are lots of permutation antichains to choose from!
How to convert antichains

• For a graph $G$, define
  \[ \Pi(G) = \{ \text{permutations } \pi : G_\pi \cong G \} \].

• **Four geometrical symmetries** give the same graph.
  (N.B. Permutations not always distinct, e.g. 54321)

• Given a permutation antichain
  \[ A = \{ \alpha_1, \alpha_2, \ldots \} \],
  want each $\Pi(G_{\alpha_i})$, to contain as few permutations as possible.

• **Fact:** $G_{\alpha_i} \not\cong G_{\alpha_j}$ iff $\sigma \not\leq \alpha_j$ for all $\sigma \in \Pi(G_{\alpha_i})$.

• So for each $\sigma \in \Pi(G_{\alpha_i})$, it suffices to find $\tau \leq \sigma$ such that $\tau \not\leq \alpha_j$ for every $j$. 
Three permutation antichains required

The diagram shows a grid with the size of the forbidden clique \( n \) on the vertical axis and the size of the forbidden path \( m \) on the horizontal axis. The grid contains filled circles and open circles, indicating the presence or absence of certain configurations. The filled circles represent the configurations that are allowed, while the open circles represent the configurations that are forbidden.
A $P_7, K_5$-free antichain

An antichain in $\text{Av}(54321, 2416375, 3152746)$ [Murphy, 2003]

For every $\pi$ in the above antichain:

- $\Pi(G_\pi)$ contains only the four symmetries.
- $\pi^{-1}$ contains 51423, but $\pi$ does not.
- Other two non-trivial symmetries are similar.
The other two antichains

$P_6, K_6$-free permutation graphs [B., 2012]

$P_7, K_4$-free permutation graphs [Murphy & Vatter, 2003]
**Wqo classes**

- **Known:** $P_m, K_3$-free permutation graphs are wqo [Lozin and Mayhill, 2011]
Wqo classes

- **Known:** $P_m, K_3$-free permutation graphs are wqo [Lozin and Mayhill, 2011]
- **Todo:** $P_5, K_n$-free permutation graphs are wqo, for all $n$. 
Here's an antichain element

- This antichain needs arbitrarily large cliques.

- $P_5, K_{126923785921975}$-free permutation graphs are wqo, but $P_5$-free permutation graphs are not wqo.
The permutation problem

Theorem

The class of permutations $Av(n \cdots 21, 24153, 31524)$ is wqo.

- $G_{n \cdots 21} \cong K_n$
- $G_{24153} \cong G_{31524} \cong P_5$ (and these are the only two permutations).
- So $Av(n \cdots 21, 24153, 31524)$ corresponds to $P_5, K_n$-free permutation graphs.
- $\sigma \leq \pi$ implies $G_\sigma \leq G_\pi$, so:

Corollary

The class of $P_5, K_n$-free permutations graphs is wqo.
• Structural characterisation of $\text{Av}(n \cdots 21, 24153, 31524)$.

**Theorem (Albert, Ruškuc, Vatter, 2014+)**

*If the *simple permutations* in a class are *geometrically griddable*, then the class is wqo.*

• Simple permutations = ‘building blocks’ of the class

• Geometrically griddable = can draw on a picture like this:

```
\ / \\
/  / \\
/ / \\
\ / \\
```
Proposition

The *simple permutations* of $\text{Av}(n \cdots 21, 24153, 31524)$ are griddable.

- Induction on $n$.
- N.B. griddable *not* geometrically griddable (this will mean nothing to you)

(Geometric) griddings

$$\text{GGrid}(\begin{array}{c} \hline \hline \end{array}) = \text{Av}(2143, 2413, 3142, 3412)$$

is a subclass of:

$$\text{Grid}(\begin{array}{c} \hline \hline \end{array}) = \text{Av}(2143, 3412)$$
Step 2 – refine the gridding

Proposition

The simple permutations of $Av(n \cdots 21, 24153, 31524)$ are griddable without NW corners.

No NW corners $=$ no cycles

- No cycles: griddable $=$ geometrically griddable.
- Now $Av(n \cdots 21, 24153, 31524)$ is wqo.
Step 2 – refine the gridding

Proposition

The simple permutations of $Av(n \cdots 21, 24153, 31524)$ are griddable without NW corners.

Proof Idea

- Take the gridding from Step 1, and look for bad rectangles.
- No two in a cell, so number is bounded:

  ![Diagram showing a gridding with bad rectangles]

- Chop each bad rectangle, to make a bigger (but still finite) gridding.
• Three classes remain: $\{P_6, K_5\}$, $\{P_6, K_4\}$ and $\{P_7, K_4\}$.
• Not griddable (in the sense used here)
• None of our antichain construction tricks work
Main reference: