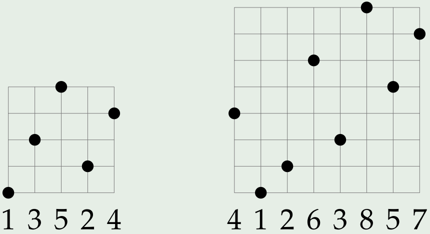
Staircases, dominoes, and the growth rate of Av(1324)

Robert Brignall

Joint work with David Bevan, Andrew Elvey Price and Jay Pantone

St Andrews, 25th April 2017

Permutation containment 101



- Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$
- Pattern containment: $\sigma \leq \pi$ if there exists a subsequence of $\pi(1) \cdots \pi(n)$ with the same relative ordering as σ.
- Containment is a partial order.
- Conversely, π avoids σ if $\sigma \not\leq \pi$.

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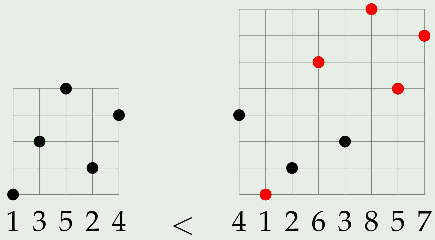

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Permutation containment 102

The Open University

Permutation class: a hereditary collection $\mathcal{C}$, i.e.

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

'Principal' classes characterised by avoiding one permutation:

$$\mathcal{C} = \mathrm{Av}(\beta) = \{\text{permutations } \pi : \beta \not\leq \pi\}.$$

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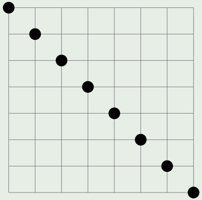
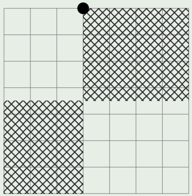


$\mathrm{Av}(12) = \{1, 21, 321, \ldots\}$ has 1 permutation of each length.



$\mathrm{Av}(132)$ has $1, 2, 5, 14, 42, \ldots$ of lengths $n = 1, 2, 3, 4, 5, \ldots$.

Counting…

...precisely

Generating function for a class $\mathcal{C}$ is the formal power series

$$f_{\mathcal{C}}(z) = \sum_{\pi \in \mathcal{C}} z^{|\pi|} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n,$$

where $\mathcal{C}_n = \{\pi \in \mathcal{C} : |\pi| = n\}$.

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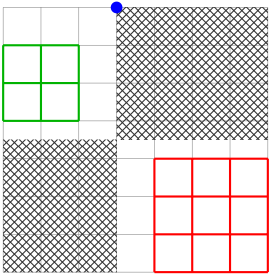
...vaguely

For principal classes $\mathrm{Av}(\beta)$, the growth rate is

$$\mathrm{gr}(\mathrm{Av}(\beta)) = \lim_{n\to\infty} \sqrt[n]{|\mathrm{Av}(\beta)_n|}.$$

Must exist due to Arratia (1999) and Marcus & Tardos (2004).

Example: $\mathcal{C} = \mathrm{Av}(132)$



Functional equation: $f_{\mathcal{C}}(z) = 1 + f_{\mathcal{C}}(z) \cdot z \cdot f_{\mathcal{C}}(z)$. Solving gives

$$f_{\mathcal{C}}(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

the generating function for the Catalan numbers $(1, 1, 2, 5, 14, 42, \ldots)$.

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From this, the growth rate follows, because:

$$\mathrm{gr}(\mathcal{C}) = \frac{1}{\sup\{r \geq 0 : f_{\mathcal{C}}(z) \text{ is analytic in } |z| < r\}}$$
$$= 4.$$

Diversion: Principal growth rates – a timeline

For a permutation β of length k:

- Stanley & Wilf (1980s): Conjecture there exists c such that

$$|\mathrm{Av}(\beta)_n| \leq c^n.$$

- Arratia (1999): Stanley-Wilf equivalent to existence of $\mathrm{gr}(\mathrm{Av}(\beta))$. He conjectures $c \leq (k-1)^2$.

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- Marcus & Tardos (2004): $c \le 15^{2k^4\binom{k^2}{k}}$ ($\Rightarrow$ proves Stanley-Wilf).

- Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006): $\mathrm{gr}(\mathrm{Av}(1324)) \ge 9.47$ ($\Rightarrow$ disproves Arratia).

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- Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006): $\mathrm{gr}(\mathrm{Av}(1324)) \geq 9.47$ ($\Rightarrow$ disproves Arratia).
- Fox (2013+): $c \geq 2^{k^{\theta(1)}}$ for almost all β ($\Rightarrow$ *really* disproves Arratia).

Diversion: Growth rates more generally

1. For a general (non-principal, proper) permutation class $\mathcal{C}$,

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$$

always exists – Marcus & Tardos (2004). Whether $\mathrm{gr}(\mathcal{C})$ always exists in general is not known.

2. Growth rate of a sequence $s_1, s_2, \ldots$ of positive integers is

$$\mathrm{gr}((s_n)) = \lim_{n\to\infty} \sqrt[n]{s_n}$$

if this exists. (I might inadvertently use this at some point!)

Counting Principal Classes

The Open University

State of knowledge, since 1997:

β	$\lvert \mathrm{Av}(\beta)_n \rvert$	$\mathrm{gr}(\mathrm{Av}(\beta))$
1	0	0
12	1	1
123 132	$\frac{1}{n+1}\binom{2n}{n}$	4
1342 2413	$\frac{(7n^2-3n-2)}{2}(-1)^{n-1}+3\sum_{k=2}^{n}2^{k+1}\frac{(2k-4)!}{k!(k-2)!}\binom{n-k+2}{2}(-1)^{n-k}$	8
1234 1243 1432 2143	$2\sum_{k=0}^{n}\binom{2k}{k}\binom{n}{k}^2\frac{3k^2+2k+1-n-2kn}{(k+1)^2(k+2)(n-k+1)}$	9
1324	?	?

Up to symmetries, this covers all $\mathrm{Av}(\beta)$ with $\lvert\beta\rvert \leq 4$.

Exact enumeration of Av(1324)

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Doron Zeilberger, 2004

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- More recently, Conway & Guttman (2015) computed

$$|\mathrm{Av}(1324)_{36}| = 85\,626\,551\,244\,475\,524\,038\,311\,935\,717$$

Growth rate of Av(1324)

Let $c = \mathrm{gr}(\mathrm{Av}(1324))$.

2004: Bóna	$c \leq 288$
2005: Bóna	$9 \leq c$
2006: Albert et al	$9.47 \leq c$
2012: Claesson, Jelínek & Steingrímsson	$c \leq 16$
[*corollary to a conjecture*	$c \leq 13.002$]
2014: Bóna	$c \leq 13.93$
2015: Bóna	$c \leq 13.74$
2015: Bevan	$9.81 \leq c$
2015: Conway & Guttman (estimate)	$c \approx 11.60 \pm 0.01$

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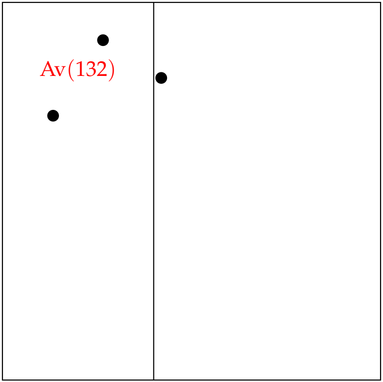
Theorem (Bevan, B., Elvey Price, Pantone)

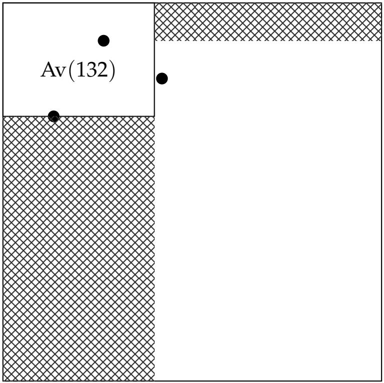
$$10.24 \leq c \leq 13.5.$$

Characterising permutations in Av(1324)



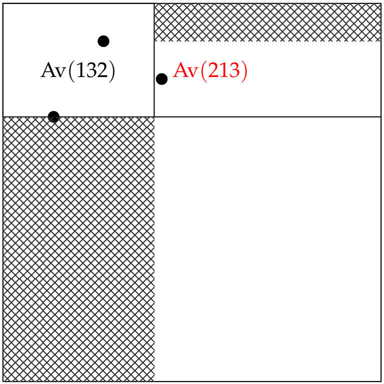
- Take any $\pi \in \text{Av}(1324)$, that does *not* avoid 132.
- Find leftmost '2' of a 132.

Characterising permutations in Av(1324)



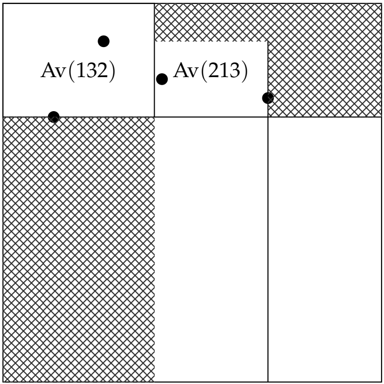
- Take any $\pi \in \mathrm{Av}(1324)$, that does *not* avoid 132.
- Shade regions where there are no points.

Characterising permutations in Av(1324)



- Take any $\pi \in \text{Av}(1324)$, that does *not* avoid 132.
- Top-right box must avoid 213.

Characterising permutations in Av(1324)



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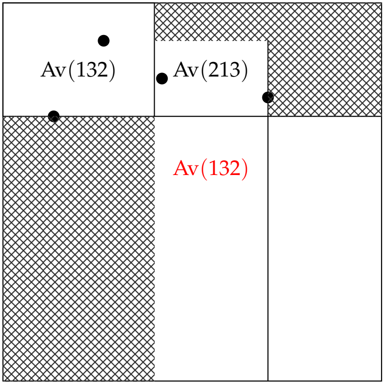

- Take any $\pi \in \mathrm{Av}(1324)$, that does *not* avoid 132.
- Middle region must avoid 132.

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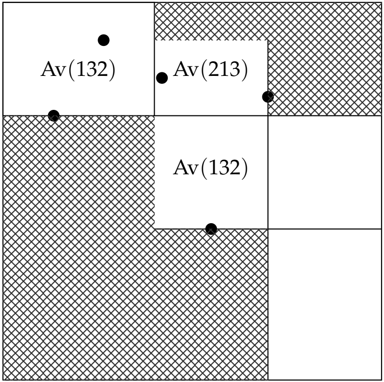

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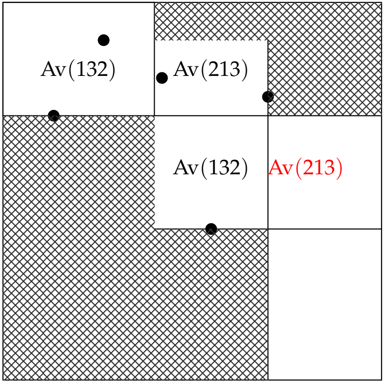

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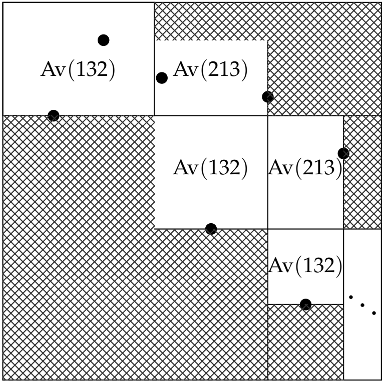
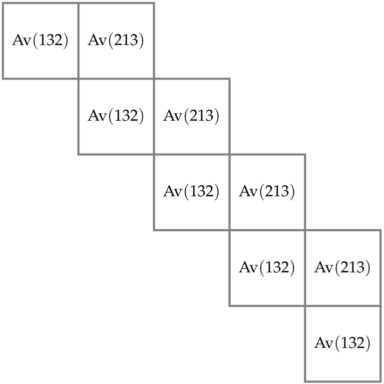


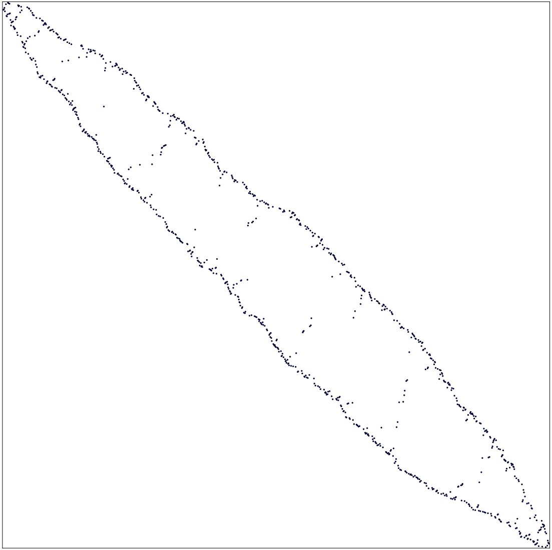
- Take any $\pi \in \mathrm{Av}(1324)$, that does *not* avoid 132.
- Shade regions, and repeat…

Characterising permutations in Av(1324)

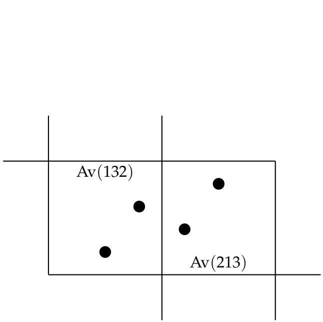


- Every $\pi \in \mathrm{Av}(1324)$ lies in such a staircase.
- Since $\mathrm{gr}(\mathrm{Av}(132)) = 4$, this gives us $\mathrm{gr}(\mathrm{Av}(1324)) \leq 16$.

Matches what a random π looks like

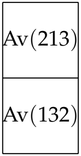


Where do I find 1324 in a staircase?



Only in two adjacent cells, and only with two points in each cell.

Domino permutations



A domino permutation:

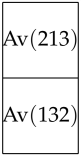
- lies in this 1×2 cell;
- comes *equipped with* a specific division into cells;
- contains no 1324, i.e. no .

Domino permutations

Theorem

The number of domino permutations on n points is $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$.

The growth rate of this sequence is $27/4$.



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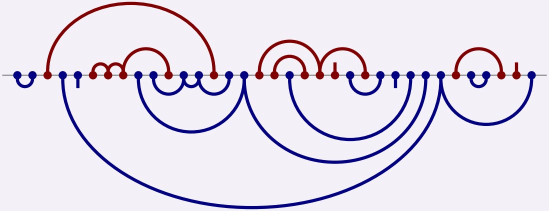

□

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Proof.

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- Functional equation (we want $C(z,0)$):

$$C(z,v) \;=\; \frac{1}{1-zC(z,v)} + z(1+v)\left(C(z,v) + \frac{C(z,v)-C(z,0)}{v}\right).$$

- Solve using iterated discriminants of Bousquet-Mélou & Jehanne (2006)

□

A familiar sequence…

The sequence given by

$$\frac{2(3n+3)!}{(n+2)!(2n+3)!}$$

is OEIS sequence A000139, and also counts, e.g.:

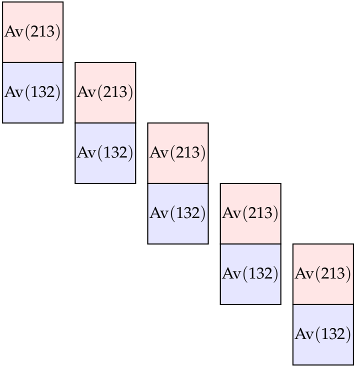
- West-2-stack-sortable permutations;
- Rooted nonseparable planar maps.

Problem

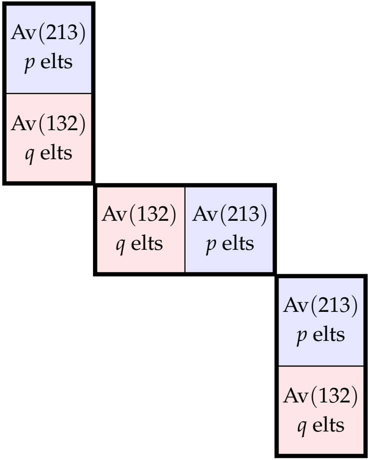
Find a bijection between domino permutations and some other combinatorial structure.

New upper bound for Av(1324)

- Take an infinite sequence of domino permutations
- Allow arbitrary interleavings between specified adjacent dominoes
- $\mathrm{gr}(\mathrm{Av}(1324)) \leq 2 \cdot \frac{27}{4} = 13.5.$

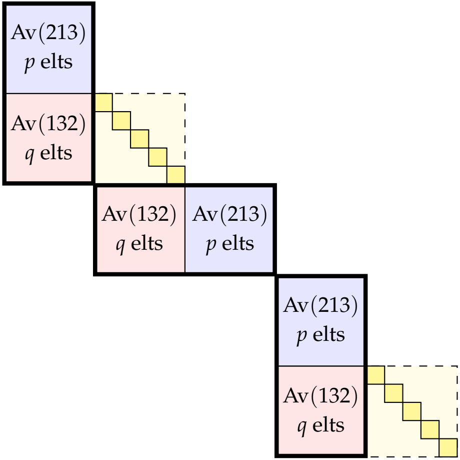


New lower bound for Av(1324)



- Begin with some dominoes and their symmetries.

New lower bound for Av(1324)



- Interleave with skew indecomposable components.

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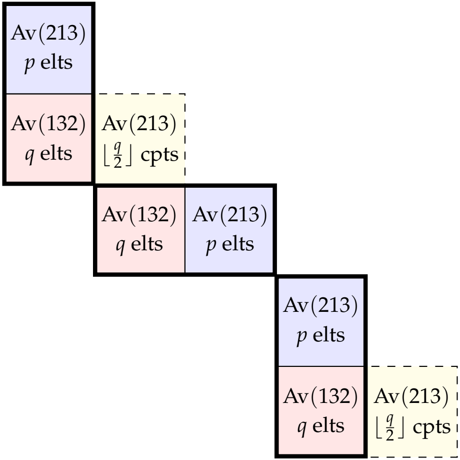

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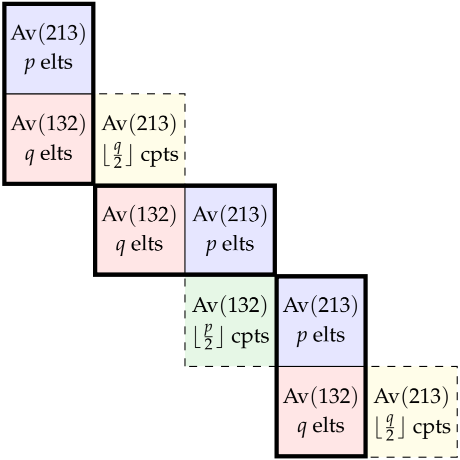

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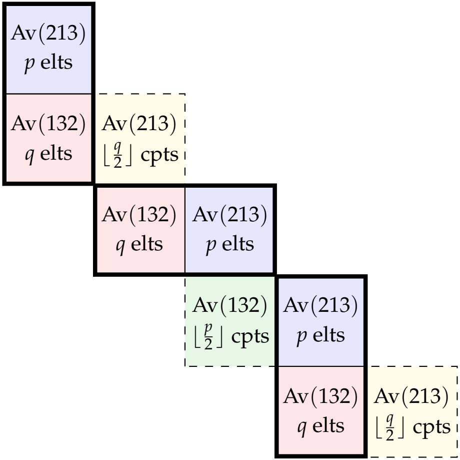
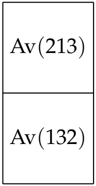
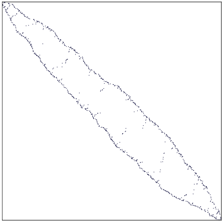


- Analysis gives $\mathrm{gr}(\mathrm{Av}(1324)) \geq 10.125$.

Leaves of a domino

Leaves: Left-to-right minima of lower cell, right-to-left maxima of upper cell.





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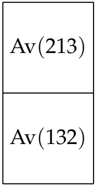
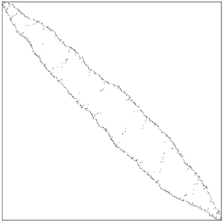


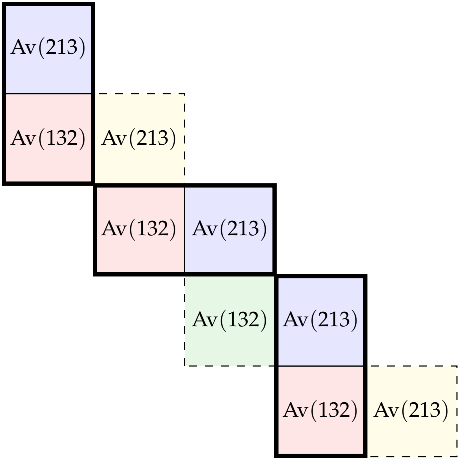


Theorem

The expected number of leaves in a domino permutation on n points is asymptotically $5n/9$, with standard deviation $O(\sqrt{n})$.

Improving the interleavings

- The $5n/9$ leaves *can* interact with skew components of yellow/green cells.



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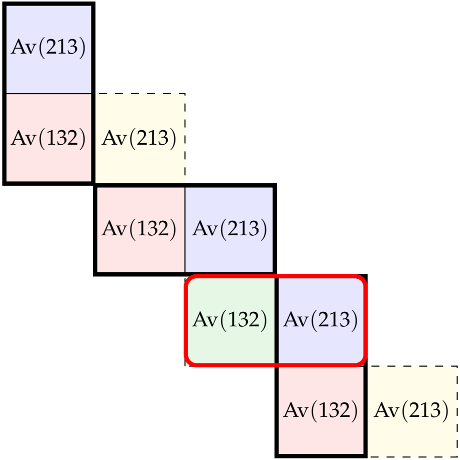

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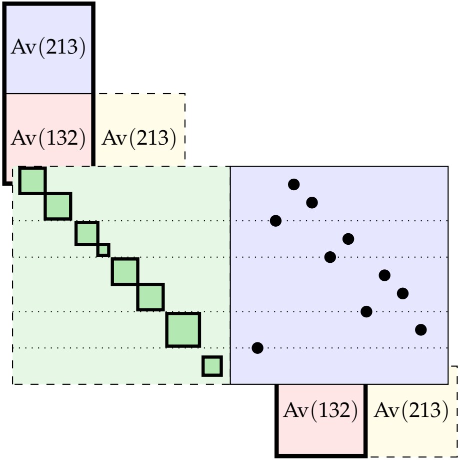
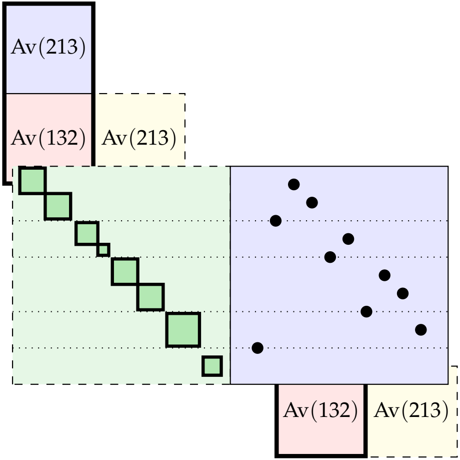


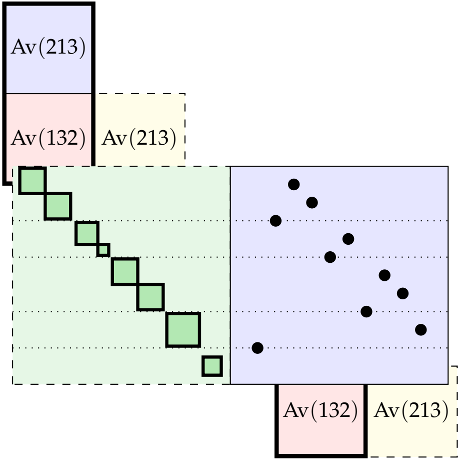
Improving the interleavings

- Interleaving in one direction gives $\mathrm{gr}(\mathrm{Av}(1324)) \geq 10.24$.



Improving the interleavings

- Interleaving in one direction gives $\mathrm{gr}(\mathrm{Av}(1324)) \geq 10.24$.
- In *both* directions we believe we get $\mathrm{gr}(\mathrm{Av}(1324)) \geq 10.271$.



Closing remarks

- Upper bound is very crude: better construction using dominoes?
- Bijection from domino permutations to something else?
- Count tri-ominoes?

   and .

  'Turning the corner' needs a new idea.

Thanks!