Staircases, dominoes, and the growth rate of \( \text{Av}(1324) \)

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St Andrews, 25th April 2017
Permutation containment 101

- Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$
- **Pattern containment**: $\sigma \leq \pi$ if there exists a subsequence of $\pi(1) \cdots \pi(n)$ with the same relative ordering as $\sigma$.
- Containment is a **partial order**.
- Conversely, $\pi$ **avoids** $\sigma$ if $\sigma \not\leq \pi$. 
Permutations in one-line notation: $\pi = \pi(1) \cdots \pi(n)$

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Containment is a partial order.

Conversely, $\pi$ avoids $\sigma$ if $\sigma \nleq \pi$. 
Permutation containment 102

**Permutation class:** a hereditary collection $\mathcal{C}$, i.e.

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$ 

‘Principal’ classes characterised by avoiding one permutation:

$$\mathcal{C} = \text{Av}(\beta) = \{\text{permutations } \pi : \beta \not\preceq \pi\}.$$
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‘Principal’ classes characterised by avoiding one permutation:

$$C = \text{Av}(\beta) = \{ \text{permutations } \pi : \beta \not\leq \pi \}.$$ 

Av(12) = \{1,21,321,\ldots\} has 1 permutation of each length. Av(132) has 1, 2, 5, 14, 42, \ldots of lengths $n = 1, 2, 3, 4, 5, \ldots$. 
Counting...

...precisely

Generating function for a class $\mathcal{C}$ is the formal power series

$$f_\mathcal{C}(z) = \sum_{\pi \in \mathcal{C}} z^{|\pi|} = \sum_{n=1}^{\infty} |\mathcal{C}_n| z^n,$$

where $\mathcal{C}_n = \{ \pi \in \mathcal{C} : |\pi| = n \}$. 
...precisely

Generating function for a class $C$ is the formal power series

$$f_C(z) = \sum_{\pi \in C} z^{|\pi|} = \sum_{n=1}^{\infty} |C_n| z^n,$$

where $C_n = \{ \pi \in C : |\pi| = n \}$.

...vaguely

For principal classes $Av(\beta)$, the growth rate is

$$\text{gr}(Av(\beta)) = \lim_{n \to \infty} \sqrt[n]{|Av(\beta)_n|}.$$  

Must exist due to Arratia (1999) and Marcus & Tardos (2004).
Example: $C = \text{Av}(132)$

Functional equation: $f_C(z) = 1 + f_C(z) \cdot z \cdot f_C(z)$. Solving gives

$$f_C(z) = \frac{1 - \sqrt{1 - 4z}}{2z},$$

the generating function for the Catalan numbers (1, 1, 2, 5, 14, 42, \ldots).
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From this, the growth rate follows, because:

$$\text{gr}(C) = \frac{1}{\sup\{r \geq 0 : f_C(z) \text{ is analytic in } |z| < r\}}$$

$$= 4.$$
Diversion: Principal growth rates – a timeline

For a permutation $\beta$ of length $k$:

- **Stanley & Wilf** (1980s): Conjecture there exists $c$ such that
  \[ |\text{Av}(\beta)_n| \leq c^n. \]

- **Arratia** (1999): Stanley-Wilf equivalent to existence of $\text{gr}(\text{Av}(\beta))$. He conjectures $c \leq (k - 1)^2$. 

- **Marcus & Tardos** (2004): $c \leq \frac{15}{2^k}$ (⇒ proves Stanley-Wilf).

- **Albert, Elder, Rechnitzer, Westcott & Zabrocki** (2006): $c \geq 2k^\theta(1)$ for almost all $\beta$ (⇒ really disproves Arratia).

- **Fox** (2013+): $c \geq 2k^\theta(1)$ for almost all $\beta$. 

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- **Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006):**
  $$\text{gr}(\text{Av}(1324)) \geq 9.47 \ (\Rightarrow \text{disproves Arratia}).$$
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- **Marcus & Tardos (2004):** $c \leq 15^2k^4\binom{k^2}{k}$ (⇒ proves Stanley-Wilf).

- **Albert, Elder, Rechnitzer, Westcott & Zabrocki (2006):** $\text{gr}(\text{Av}(1324)) \geq 9.47$ (⇒ disproves Arratia).

- **Fox (2013+):** $c \geq 2^{k^{\theta(1)}}$ for almost all $\beta$ (⇒ really disproves Arratia).
1. For a **general** (non-principal, proper) permutation class $\mathcal{C}$,

$$\overline{\text{gr}}(\mathcal{C}) = \limsup_{n \to \infty} \sqrt[n]{|\mathcal{C}_n|}$$

always exists – Marcus & Tardos (2004). Whether $\text{gr}(\mathcal{C})$ always exists in general is **not known**.

2. Growth rate of a sequence $s_1, s_2, \ldots$ of positive integers is

$$\text{gr}((s_n)) = \lim_{n \to \infty} \sqrt[n]{s_n}$$

if this exists. (I might inadvertently use this at some point!)
### Counting Principal Classes

State of knowledge, since 1997:

| \( \beta \) | \( |\text{Av}(\beta)_n| \) | \( \text{gr}(\text{Av}(\beta)) \) |
|---|---|---|
| 1 | 0 | 0 |
| 12 | 1 | 1 |
| 123 | \( \frac{1}{n+1} \binom{2n}{n} \) | 4 |
| 132 | \( \frac{1}{n+1} \binom{2n}{n} \) | 4 |
| 1342 | \( \frac{7n^2 - 3n - 2}{2} (-1)^{n-1} + 3 \sum_{k=2}^{n} 2^{k+1} \frac{(2k-4)!}{k!(k-2)!} \frac{n-k+2}{2} (-1)^{n-k} \) | 8 |
| 2413 | \( \frac{3k^2 + 2k + 1 - n - 2kn}{(k+1)^2(k+2)(n-k+1)} \) | 9 |
| 1234 | \( ? \) | ? |
| 1243 | \( ? \) | ? |
| 1432 | \( ? \) | ? |
| 2143 | \( ? \) | ? |
| 1324 | \( ? \) | ? |

Up to symmetries, this covers all \( \text{Av}(\beta) \) with \( |\beta| \leq 4 \).
Exact enumeration of $\text{Av}(1324)$

“Not even God knows $|\text{Av}(1324)_{1000}|$."

Doron Zeilberger, 2004
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• More recently, Conway & Guttman (2015) computed

$$|\text{Av}(1324)_{36}| = 85\,626\,551\,244\,475\,524\,038\,311\,935\,717$$
Growth rate of $\text{Av}(1324)$

Let $c = \text{gr}(\text{Av}(1324))$.

2004: Bóna $c \leq 288$
2005: Bóna $9 \leq c$
2006: Albert et al $9.47 \leq c$
2012: Claesson, Jelínek & Steingrímsson $c \leq 16$
2014: Bóna $c \leq 13.93$
2015: Bóna $c \leq 13.74$
2015: Bevan $9.81 \leq c$
2015: Conway & Guttman (estimate) $c \approx 11.60 \pm 0.01$
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2012: Claesson, Jelínek & Steingrímsson \( c \leq 16 \)
[corollary to a conjecture \( c \leq 13.002 \)]
2014: Bóna \( c \leq 13.93 \)
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**Theorem (Bevan, B., Elvey Price, Pantone)**

\[ 10.24 \leq c \leq 13.5. \]
Characterising permutations in $\text{Av}(1324)$

- Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
- Find leftmost ‘2’ of a 132.
Characterising permutations in $\text{Av}(1324)$

- Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
- Shade regions where there are no points.
Characterising permutations in $\text{Av}(1324)$

- Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
- Top-right box must avoid 213.
• Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
• Shade regions where there are no points.
• Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
• Middle region must avoid 132.
• Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
• Shade regions where there are no points.
• Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
• Middle region must avoid 213.
• Take any $\pi \in \text{Av}(1324)$, that does not avoid 132.
• Shade regions, and repeat...
Characterising permutations in $\text{Av}(1324)$

- Every $\pi \in \text{Av}(1324)$ lies in such a staircase.
- Since $\text{gr}(\text{Av}(132)) = 4$, this gives us $\text{gr}(\text{Av}(1324)) \leq 16$. 
Matches what a random $\pi$ looks like
Where do I find 1324 in a staircase?

Only in two adjacent cells, and only with two points in each cell.
Domino permutations

Theorem

The number of domino permutations on $n$ points is

$$2 \left(3^n + 3\right)! \left(n + 2\right)! \left(2n + 3\right)!.$$
Theorem

The number of domino permutations on $n$ points is \( \frac{2(3n + 3)!}{(n + 2)!(2n + 3)!} \).

The growth rate of this sequence is $27/4$.

A domino permutation:
- lies in this $1 \times 2$ cell;
- comes equipped with a specific division into cells;
- contains no 1324, i.e. no \( \vdash \).
Theorem

The number of domino permutations on $n$ points is
\[ \frac{2(3n + 3)!}{(n + 2)!(2n + 3)!}. \]

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Proof.

- Bijection with arch configurations having no \( - - \).
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The number of domino permutations on \( n \) points is \( \frac{2(3n + 3)!}{(n + 2)!(2n + 3)!} \).

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Proof.

- Bijection with arch configurations having no \( \circ \).
- Functional equation (we want \( C(z, 0) \)):

\[
C(z, v) = \frac{1}{1 - zC(z, v)} + z(1 + v) \left( C(z, v) + \frac{C(z, v) - C(z, 0)}{v} \right).
\]

The sequence given by
\[
\frac{2(3n + 3)!}{(n + 2)!(2n + 3)!}
\]

is OEIS sequence A000139, and also counts, e.g.:

- West-2-stack-sortable permutations;
- Rooted nonseparable planar maps.

**Problem**

Find a bijection between domino permutations and some other combinatorial structure.
New upper bound for $\text{Av}(1324)$

- Take an infinite sequence of domino permutations
- Allow **arbitrary interleavings** between specified adjacent dominoes
- $\text{gr}(\text{Av}(1324)) \leq 2 \cdot \frac{27}{4} = 13.5.$
New lower bound for $\text{Av}(1324)$

- Begin with some dominoes and their symmetries.
New lower bound for $\text{Av}(1324)$

- Interleave with skew indecomposable components.
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New lower bound for $\text{Av}(1324)$

- Analysis gives $\text{gr}(\text{Av}(1324)) \geq 10.125$. 
Leaves of a domino

**Leaves**: Left-to-right minima of lower cell, right-to-left maxima of upper cell.

\[
\begin{align*}
\text{Av}(213) \\
\text{Av}(132)
\end{align*}
\]
Leaves of a domino

Leaves: Left-to-right minima of lower cell, right-to-left maxima of upper cell.

Theorem

The expected number of leaves in a domino permutation on n points is asymptotically $5n/9$, with standard deviation $O(\sqrt{n})$. 
The $5n/9$ leaves can interact with skew components of yellow/green cells.
Improving the interleavings

- The $5n/9$ leaves *can* interact with skew components of yellow/green cells.
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Improving the interleavings

- Interleaving in one direction gives $\text{gr}(\text{Av}(1324)) \geq 10.24$. 
Improving the interleavings

- Interleaving in one direction gives $\text{gr}(\text{Av}(1324)) \geq 10.24$.
- In both directions we believe we get $\text{gr}(\text{Av}(1324)) \geq 10.271$. 
Closing remarks

• Upper bound is very crude: better construction using dominoes?

• Bijection from domino permutations to something else?

• Count tri-ominoes? □□ and □. ‘Turning the corner’ needs a new idea.
Thanks!