Grid Classes

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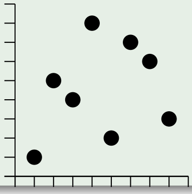
Outline

Setting the Scene

- Permutation of length n: an ordering on the symbols $1, \ldots, n$.
- For example: $\pi = 15482763$.
- Graphical viewpoint: plot the points $(i, \pi(i))$.

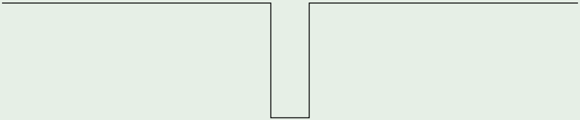
Example



Stack Sorting

- Knuth (1969): what permutations can be sorted through a stack?

Example



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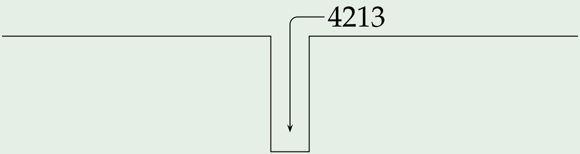

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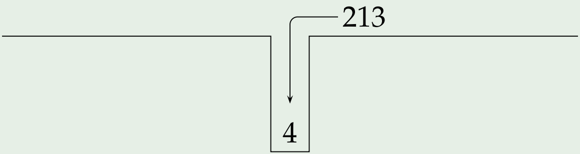

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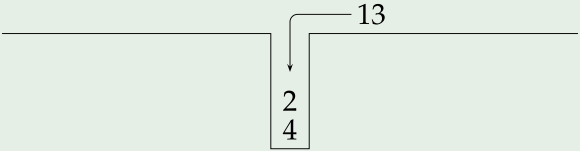

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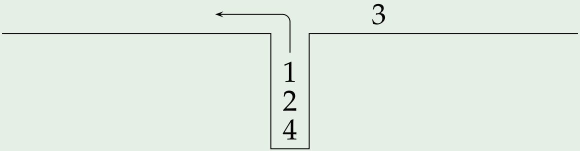

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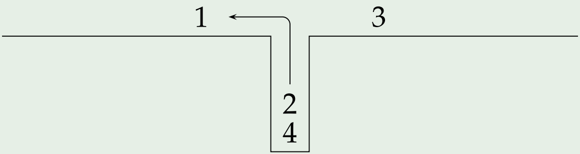

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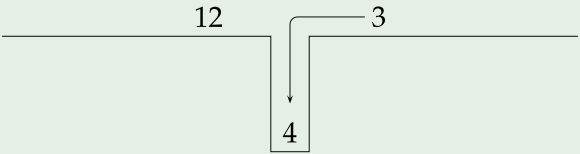

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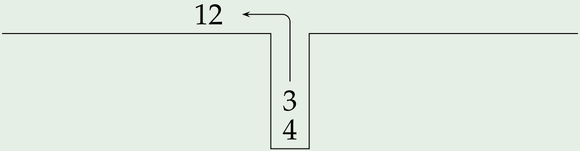

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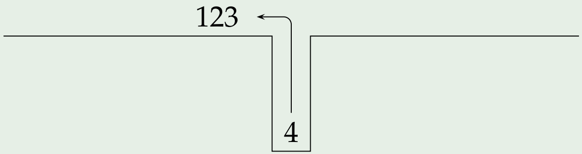

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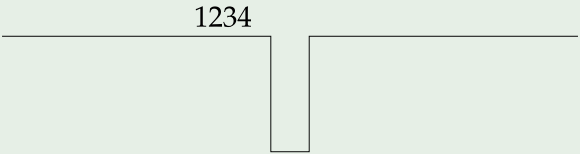

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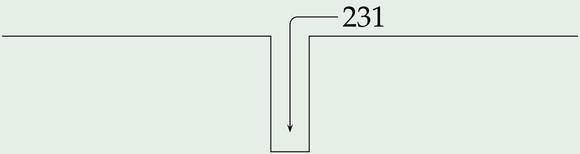

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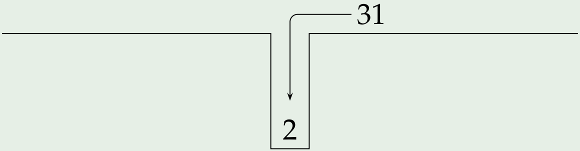

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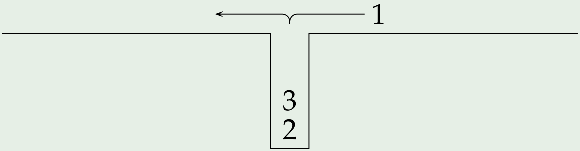

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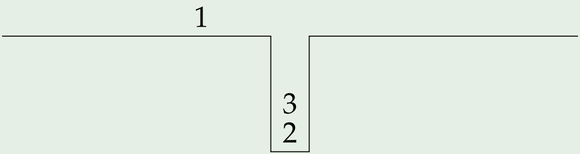

- 231 is not stack-sortable.

Stack Sorting

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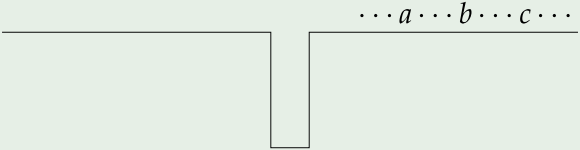
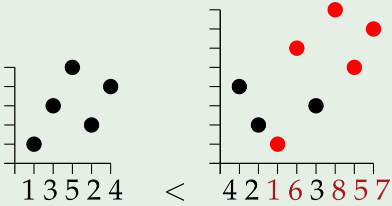


- 231 is not stack-sortable.
- In general: can't sort any permutation with a subsequence *abc* such that $c < a < b$. (*abc* forms a 231 "pattern".)

Permutation Containment

- Write permutations in one-line notation, e.g. $\tau = 13524$.
- A permutation $\tau = \tau(1) \cdots \tau(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ if there exists a subsequence $\sigma(i_1)\sigma(i_2) \cdots \sigma(i_k)$ order isomorphic to τ.

Example



Permutation Classes

- Containment is a partial order on the set of all permutations.
- Recall: downsets are permutation classes. i.e. $\pi \in \mathcal{C}$ and $\sigma \leq \pi$ implies $\sigma \in \mathcal{C}$.
- Each class has a unique set of minimal forbidden elements. Write

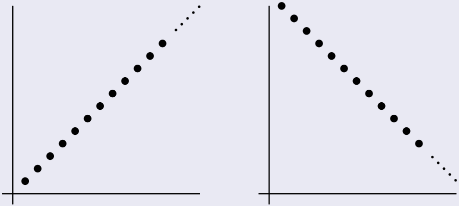
$$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

B is (unfortunately) called the basis.

Easy Examples

- $\text{Av}(21) = \{1, 12, 123, 1234, \ldots\}$, the increasing permutations.
- $\text{Av}(12) = \{1, 21, 321, 4321, \ldots\}$, the decreasing permutations.

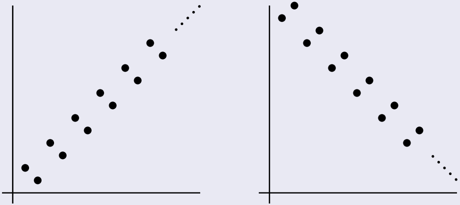
Typical Elements



Easy Examples

- $\oplus 21 = \text{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots\}$.
- $\ominus 12 = \text{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots\}$.

Typical Elements



Questions

Given a permutation class $\mathcal{C}$:

- Basis: $\mathcal{C} = \text{Av}(B)$ for some B. Is B finite?
- Well-quasi-order: Does $\mathcal{C}$ contain infinite antichains?
- Structure: What do the permutations in $\mathcal{C}$ look like?
- Enumeration: How many of length n? Asymptotics?

Exact Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.
- $\sum |\mathcal{C}_n| x^n$ is the generating function.

Example

The generating function of $\mathcal{C} = \text{Av}(12)$ is:

$$1 + x + x^2 + x^3 + \cdots \quad = \quad \frac{1}{1-x}$$

Exact Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.
- $\sum |\mathcal{C}_n| x^n$ is the generating function.

Example

The generating function of $\oplus 21 = \mathrm{Av}(231, 312, 321)$ is:

$$1 + x + 2x^2 + 3x^3 + \cdots \quad = \quad \frac{1}{1 - x - x^2}$$

Asymptotic Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.

Theorem (Marcus and Tardos, 2004)

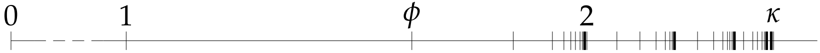
For every permutation class $\mathcal{C}$ other than the class of all permutations, there exists a constant K such that

$$\limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} \leq K.$$

- Big open question: does the growth rate, $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$, always exist?

Small Growth Rates

- Growth rate of $\mathcal{C}$ is $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ (if it exists).
- Below $\kappa \approx 2.20557$, growth rates exist and can be characterised [Vatter, 2011]:

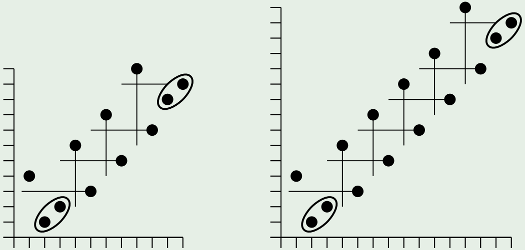


- κ is the lowest growth rate where we encounter infinite antichains, and hence uncountably many permutation classes.
- The proof of this uses grid classes.

Increasing Oscillations: an Infinite Antichain

- (Infinite) set of pairwise incomparable permutations.

Two typical elements



- Need to show there is no embedding of one in the other.

Increasing Oscillations: an Infinite Antichain

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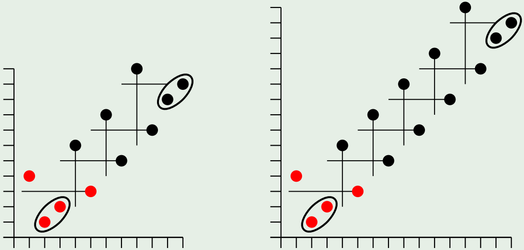

- Anchor: bottom copies of 4123 must match up.

Increasing Oscillations: an Infinite Antichain

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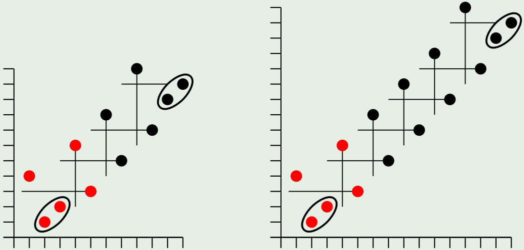
Two typical elements



- Each point is matched in turn.

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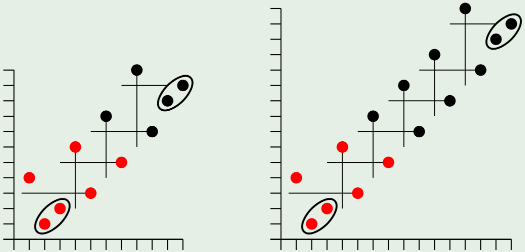

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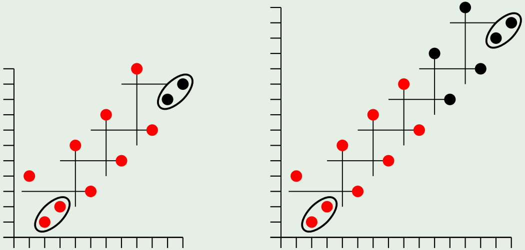

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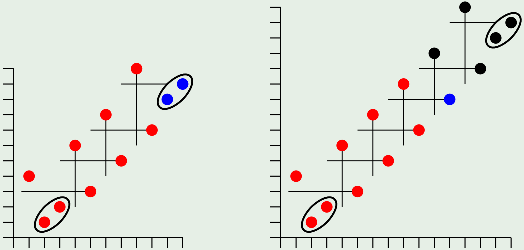

- Last pair cannot be embedded.

Increasing Oscillations: an Infinite Antichain

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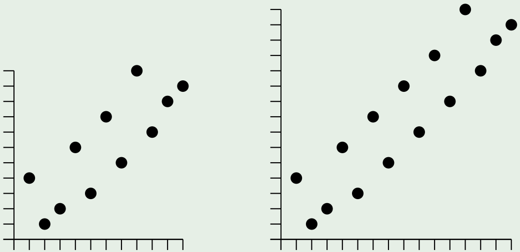

- Alternatively, make a graph: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$

Increasing Oscillations: an Infinite Antichain

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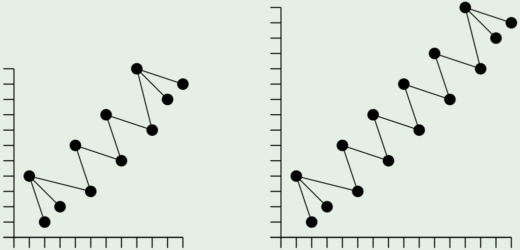
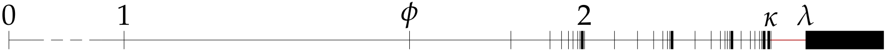


- Neither is the induced subgraph of the other.

Increasing Oscillations are Important

- At $\kappa \approx 2.20557$, we find permutation classes that contain the increasing oscillating antichain.
- Above $\lambda \approx 2.48188$, every real number is the growth rate of a permutation class [Vatter, 2010].
  The proof builds classes based on this antichain.



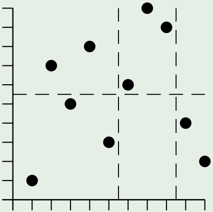
- From order to chaos: What lies between κ and λ?

Grid Classes

- Idea: describe complicated classes in terms of easier ones.
- Matrix $\mathcal{M}$ whose entries are (infinite) permutation classes.
- $\mathrm{Grid}(\mathcal{M})$ the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(231) & \emptyset \\ \mathrm{Av}(123) & \emptyset & \mathrm{Av}(12) \end{pmatrix}$.



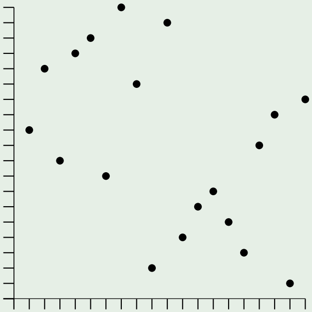
$\in \mathrm{Grid}(\mathcal{M})$

Monotone Grid Classes

- Special case: all cells of $\mathcal{M}$ are Av(21) or Av(12).
- Rewrite $\mathcal{M}$ as a matrix with entries in $\{0, 1, -1\}$.

Example

$$\mathcal{M} = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix}$$



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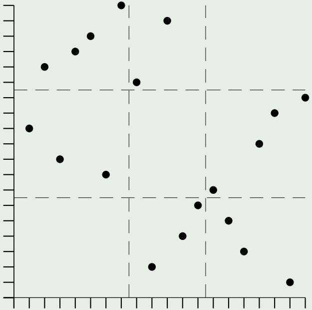


Basis of Grid Classes

Question

Given a grid class $Grid(\mathcal{M})$*, what is its basis? (Is it finite?)*

- A complete answer to this question seems a very long way off...

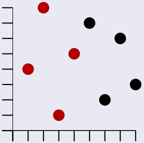
Juxtapositions: $1 \times k$ grids

Lemma (Atkinson, 1999)

Grid$(\mathcal{C}\ \mathcal{D})$ is finitely based if $\mathcal{C}$ and $\mathcal{D}$ are finitely based.

Proof.

Basis elements formed by gluing basis elements of $\mathcal{C}$ and $\mathcal{D}$ together:



- Red: Basis element of $\mathcal{C}$.

□

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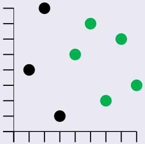

- Green: Basis element of $\mathcal{D}$, overlaps by (at most) 1 with red.

□

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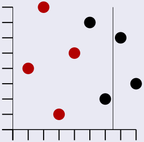

- Can we grid it? If line too far right: LHS is bad.

□

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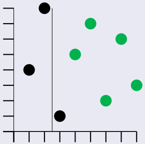

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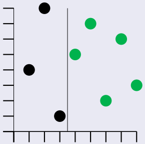

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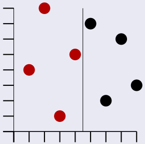


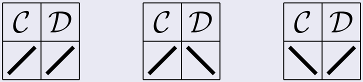
- Crossover point: permutation not in $\mathrm{Grid}(\mathcal{C}\ \mathcal{D})$.

□

Basis: 2×2 Grids

Lemma (Albert, Atkinson, B., 2010)

The grid classes



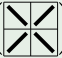
are finitely based, for finitely based classes $\mathcal{C}$ and $\mathcal{D}$.

- Proof: same kind of arguments to 1×2 case.
- Does not obviously extend to $2 \times k$.

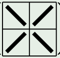
Geometric Grid Classes

- Fill a square grid with 45° lines.
- Make permutations by choosing points from these lines.
- These are not just monotone grid classes:

Example

GGrid() $= \mathrm{Av}(2143, 2413, 3142, 3412)$

is a subclass of:

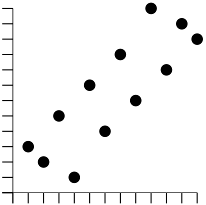
Grid() $= \mathrm{Av}(2143, 3412)$

Theorem (Albert, Atkinson, Bouvel, Ruškuc, Vatter, 2010)

Every geometric grid class is finitely based.

Basis: Some final comments

- Strong belief that all monotone grid classes are finitely based. (Not just geometric ones.)
- Grid $\begin{pmatrix} \emptyset & \text{Av}(321654) \\ \text{Av}(321654) & \emptyset \end{pmatrix}$ is not finitely based:

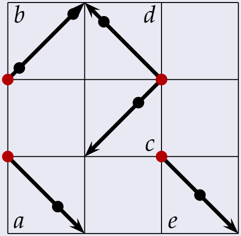


More geometry

Theorem (Albert et al)

Geometric grid classes can be encoded by a regular language, and therefore have rational generating functions.

Proof.



4 1 6 5 3 2 $\mapsto$

(Homage to Nik Ruškuc for the illustration.) □

Practical enumeration

- Test ground: count classes avoiding two permutations of length 4.
- Up to symmetry, four we can use this on:

$$\mathrm{Av}(1324, 4312) \qquad \mathrm{Av}(2143, 4231)$$

$$\mathrm{Av}(2143, 4312) \qquad \mathrm{Av}(2143, 4321)$$

- Each class is the union of several geometric grid classes.

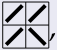
Enumerating Av(2143, 4312)

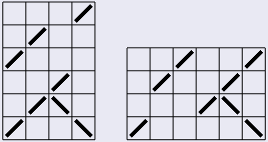
Theorem (Albert, Atkinson, B., 2011)

Av(2143, 4312) *has generating function*

$$\frac{1 - 13x + 69x^2 - 191x^3 + 294x^4 - 252x^5 + 116x^6 - 23x^7}{(1-x)^2(1-3x)^2(1-3x+x^2)^2}$$

Proof.

This class is contained in , and so is the union of:



Enumeration of these classes is fiddly… □

Enumerating Av(2143, 4231)

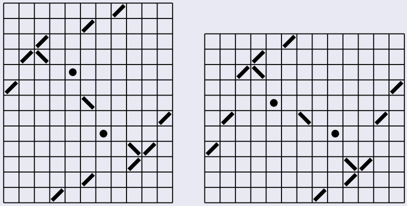
Theorem (Albert, Atkinson, B., 2010)

$Av(2143, 4231)$ has generating function

$$\frac{1 - 12x + 60x^2 - 162x^3 + 259x^4 - 252x^5 + 146x^6 - 46x^7 + 8x^8}{(1-x)^4(1-3x)(1-3x+x^2)^2}$$

Proof.

This class is the union of:



□

Well-quasi-order

Recall: well-quasi-order = no infinite antichains.

Theorem (Vatter and Waton, 2007)

Geometric grid classes are well-quasi-ordered.

Proof.

- Geometric grid classes can be encoded by words over a finite alphabet.
- Words are wqo by Higman's Lemma.

□

The Graph of a Matrix

- Graph of a matrix, $G(\mathcal{M})$, formed by connecting together all non-zero entries that share a row or column and are not "separated" by any other nonzero entry.

Example

$$\begin{pmatrix} \mathcal{C} & 0 & 0 & \mathcal{D} \\ 0 & 0 & \mathcal{E} & 0 \\ \mathcal{D} & \mathcal{E} & 0 & \mathcal{C} \\ 0 & 0 & 0 & \mathcal{D} \end{pmatrix}$$

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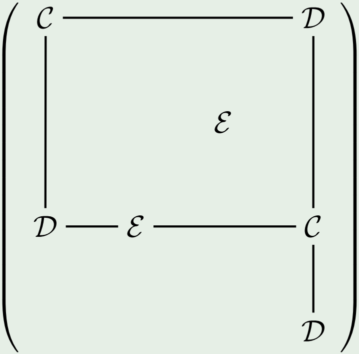


When monotone = geometric

- For a monotone gridding matrix $\mathcal{M}$:

Lemma (Albert et al)

$GGrid(\mathcal{M}) = Grid(\mathcal{M})$ if and only if the graph of $\mathcal{M}$ is a forest.

- Proof idea: you can "iron out" kinks in the lines when there are no cycles.

Corollary

Monotone grid classes of forests are well-quasi-ordered.

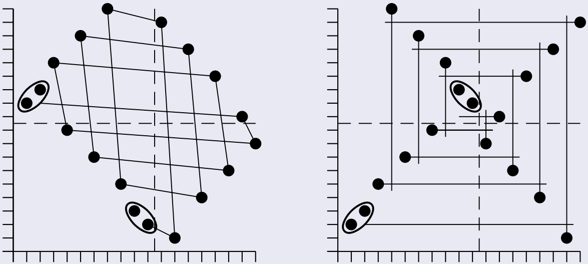
Monotone grids and well-quasi-order

Theorem (Murphy and Vatter, 2003)

The monotone grid class $\text{Grid}(\mathcal{M})$ *is wqo if and only if* $G(\mathcal{M})$ *is a forest.*

Proof.

(⇒) Construct infinite antichains that "walk" around a cycle.



□

Griddability

- Idea: Want wqo for general permutation classes. When does this result hold?
- $\mathcal{C}$ is $\mathcal{D}$-griddable if there exists a finite matrix $\mathcal{M}$ whose entries are (subclasses of) $\mathcal{D}$, and $\mathcal{C} \subseteq \text{Grid}(\mathcal{M})$.
  Roughly, every permutation in $\mathcal{C}$ can be "chopped up" and shown to lie in $\text{Grid}(\mathcal{M})$.
- Monotone griddable: a class $\mathcal{C}$ is the subclass of a monotone grid class.

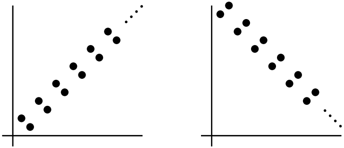
When is a class griddable?

Question

When is a class $\mathcal{C}$ monotone griddable?

Answer [Huczynska & Vatter, 2006]

A class $\mathcal{C}$ is monotone griddable if and only if it contains neither the classes $\oplus 21$ nor $\ominus 12$.



- More generally: $\mathcal{D}$-griddable classes can be characterised for any class $\mathcal{D}$ [Vatter, 2011].

Beyond monotone

- What can we say about infinite antichains for general grid classes?
- Next stage: allow cells labelled by $\oplus 21$ and $\ominus 12$.

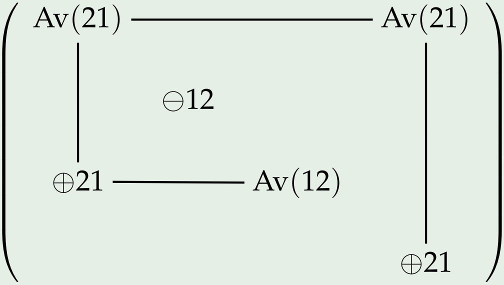
Example

$$\begin{pmatrix} \mathrm{Av}(21) & 0 & 0 & \mathrm{Av}(21) \\ 0 & \ominus 12 & 0 & 0 \\ \oplus 21 & 0 & \mathrm{Av}(12) & 0 \\ 0 & 0 & 0 & \oplus 21 \end{pmatrix}$$

Beyond monotone

- What can we say about infinite antichains for general grid classes?
- Next stage: allow cells labelled by $\oplus 21$ and $\ominus 12$.

Example



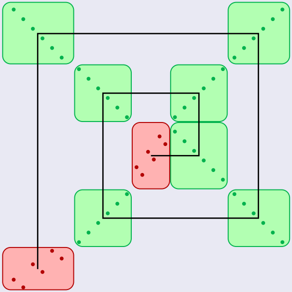
- Can assume graph is a forest, but the number of non-monotone-griddable cells in each component matters.

Two is too many

Theorem (B.)

A grid class whose graph has a component containing two or more non-monotone-griddable cells is not wqo.

Proof.



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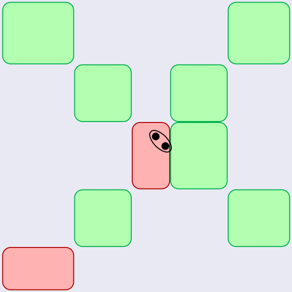

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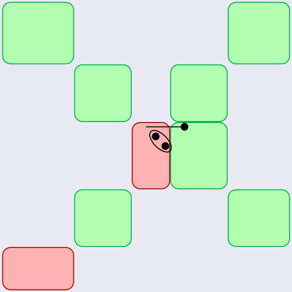

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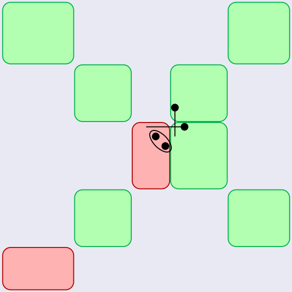

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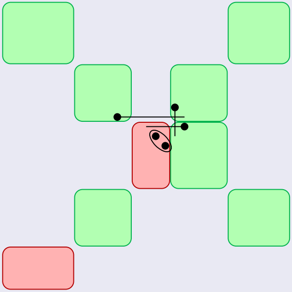

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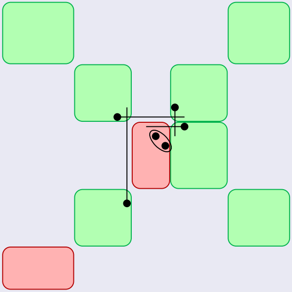

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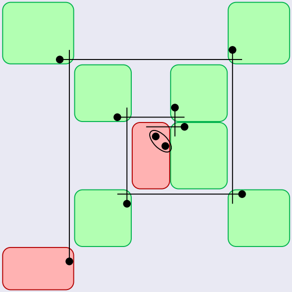

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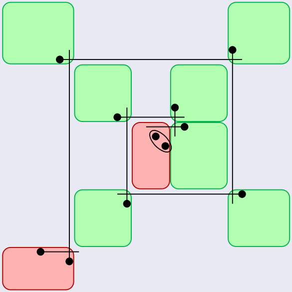

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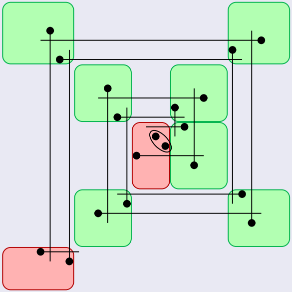

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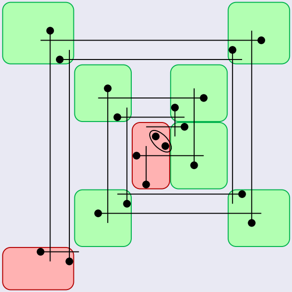

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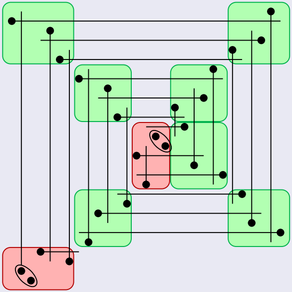


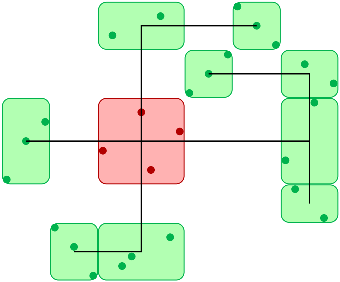
□

Just one non-monotone per component

Simple permutations are the "building blocks" of permutation classes.

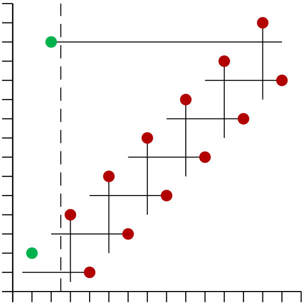
Theorem (B.)

If the non-monotone cell contains only finitely many simple permutations, then the grid class is wqo.



But sometimes one is too much...

- One cell containing arbitrarily long increasing oscillations next to a monotone cell is bad...



- Mind the gap: between finite simples and infinite oscillations, not (yet) known.

Thanks!