Finite Basis Results of Wreath Products

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Introduction

Pattern Involvement

- Regard a permutation of length n as an ordering of the symbols $1, \dots, n$.
- A permutation $\tau = t_1 t_2 \dots t_k$ is involved in the permutation $\sigma = s_1 s_2 \dots s_n$ if there exists a subsequence $s_{i_1}, s_{i_2}, \dots, s_{i_k}$ order isomorphic to τ.
- Write $\tau \preccurlyeq \sigma$.
- Example: The permutation 13524 is involved in 42163857 because of the sequence 16857.
- Involvement forms a partial order on the set of all permutations.

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Closed Classes

- A set of permutations X is closed if

$$\sigma \in X \text{ and } \tau \preccurlyeq \sigma \Rightarrow \tau \in X.$$

- Equivalently, a closed class X can be seen to avoid certain permutations.
- We may describe X in terms of its minimal avoidance set, or basis - the minimal permutations not in X. Write

$$X = Av(B)$$

to mean X has basis B.
- Basis B is not necessarily finite. If it is, then X is finitely based.

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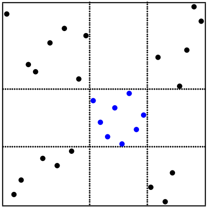
Permutation Structure

Definition

For a permutation $\sigma = s_1 s_2 \dots s_n$:

- A sequence is any set of symbols $s_{i_1}, s_{i_2}, \dots, s_{i_k}$ from σ with $i_1 < i_2 < \dots < i_k$.
- A segment is a sequence of adjacent symbols, $s_i, s_{i+1}, \dots, s_{i+j}$.
- An interval or block of σ is a segment $s_i s_{i+1} \dots s_{i+j}$, in which the set of values is contiguous:



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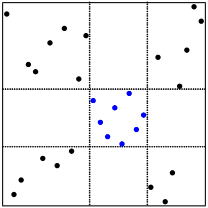

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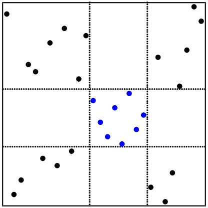

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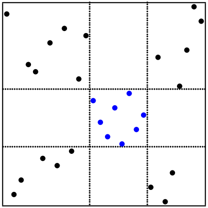

Wreath Product I

Definition

The wreath product of the set of permutations X with the set of permutations Y is the set $X \wr Y$ of permutations

$$\sigma = \alpha_1 \alpha_2 \ldots \alpha_k$$

such that:

(i) each α_i is an interval,

(ii) each α_i is order isomorphic to a permutation of Y,

(iii) if for every i we pick a symbol a_i from α_i, then $a_1 a_2 \ldots a_k$ is order isomorphic to a permutation in X.

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Wreath Product II

Example

- $X = \{1, 12\}$, $Y = \{1, 12, 21\}$:

- $X \wr Y = \{1, 12, 21, 123, 132, 213, 1234, 1243, 2134, 2143\}$.

Wreath Product II

Example

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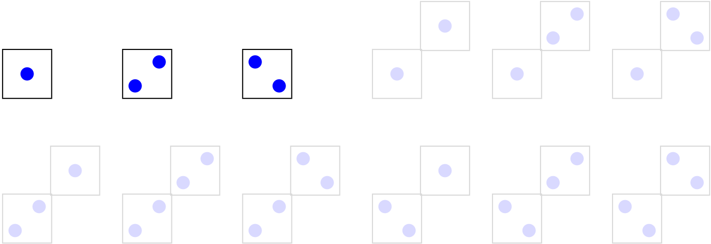

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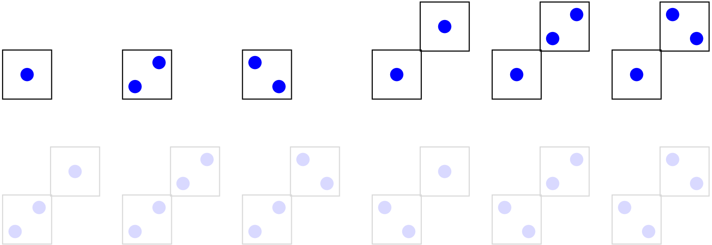

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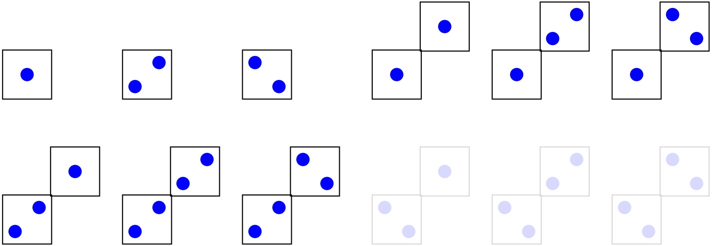

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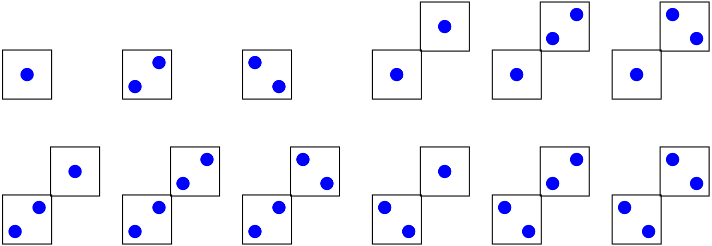

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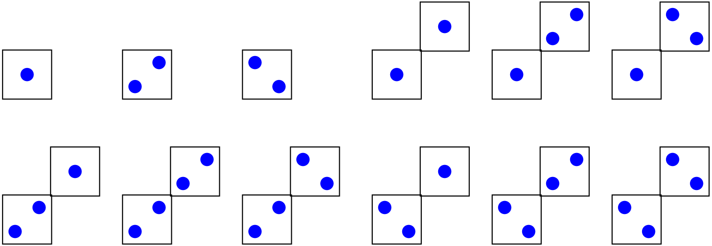

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Wreath Product III

- If X and Y are closed then $X \wr Y$ is closed.
- If X and Y are finitely based, is $X \wr Y$ finitely based?
- Not true – Atkinson proves $\mathcal{A}(21) \wr \mathcal{A}(321654)$ has infinite basis.
- Half the problem: which classes Y obey

$$X \text{ finitely based} \Rightarrow X \wr Y \text{ finitely based?}$$

Y has the Wreath Finite Basis Property (WFBP).

Example

$Y = \{1\}$. Then $X \wr Y = X$ for any class X.

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For any closed class Y, the permutation σ has Y-profile

$$\sigma^{(Y)} = s_1 s_2 \ldots s_m$$

if σ can be partitioned into segments

$$\sigma = \sigma_1 \sigma_2 \ldots \sigma_m$$

subject to

(i) each σ_i is a non-empty interval, order isomorphic to a permutation from Y,

(ii) $\sigma_i < \sigma_j$ if and only if $s_i < s_j$.

Made unique by first picking σ_1 maximally, then σ_2, then σ_3, etc.

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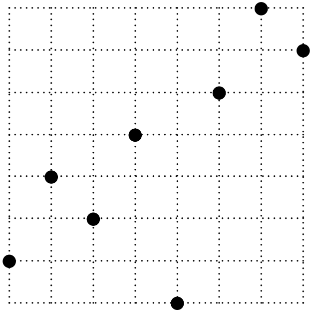

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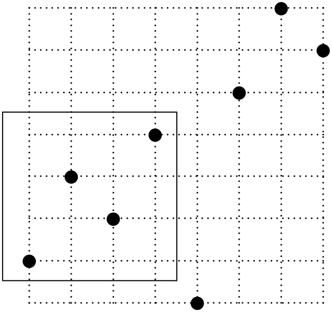

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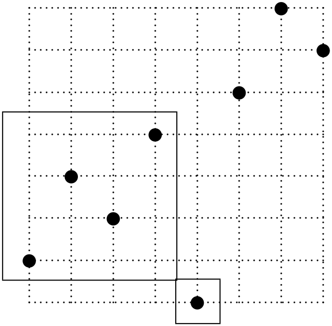

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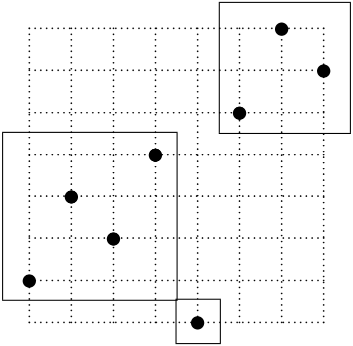

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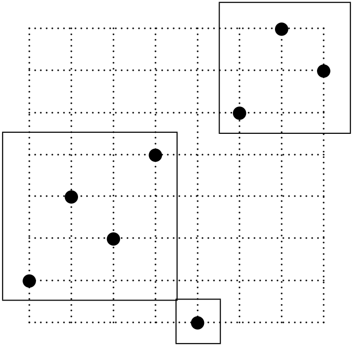

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Profiles & the Wreath Product

Theorem

For any closed classes X and Y,

$$\sigma \in X \wr Y \iff \sigma^{(Y)} \in X.$$

- Allows us to jump between the classes $X \wr Y$ and X.
- The reverse is also true (and more useful):

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Existing Approaches to WFBP

Atkinson, Restricted Permutations and the Wreath Product, 2002.

Theorem

The pattern class $I = \{1, 12, 123, 1234, \ldots\} = \mathrm{Av}(\beta)$ *of increasing permutations possesses the WFBP.*

Proof.

- Any basis element β of $X \wr I$ can be constructed from a basis element of X.
- The construction algorithm ensures $|\beta|$ is bounded in terms of the length of basis elements of X.
- Basis elements of X have bounded size since X is finitely based, so $|\beta|$ is also of bounded size.

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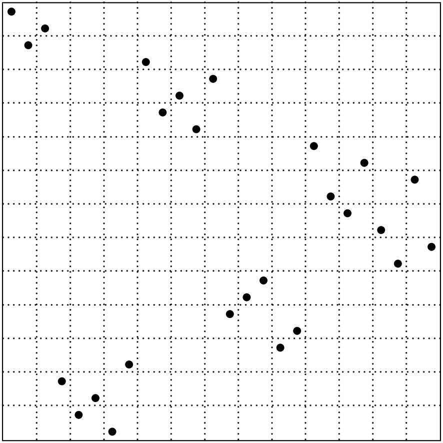
- Basis elements of X have bounded size since X is finitely based, so $|\beta|$ is also of bounded size.

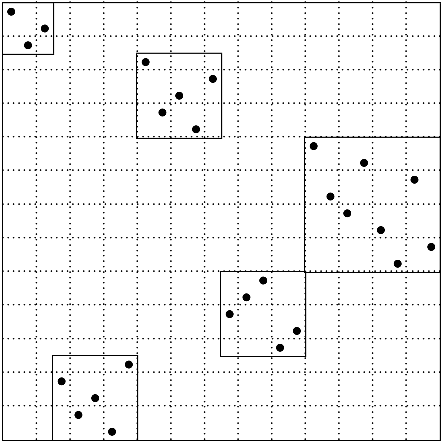
□

Generalised Approach to WFBP



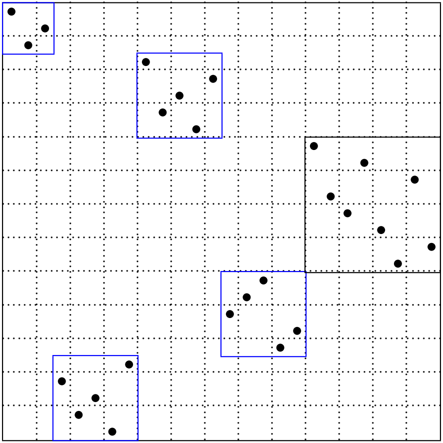
- Basis element β of $X \wr Y$.

Generalised Approach to WFBP



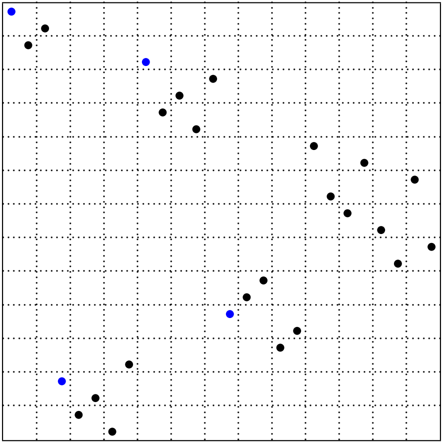
- Take the Y-profile ($\beta^{(Y)} = 51423$).

Generalised Approach to WFBP



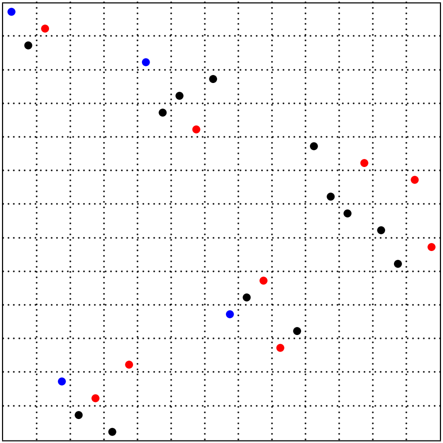
- Find basis element $\gamma = 4132$ of X inside $\beta^{(Y)}$.

Generalised Approach to WFBP



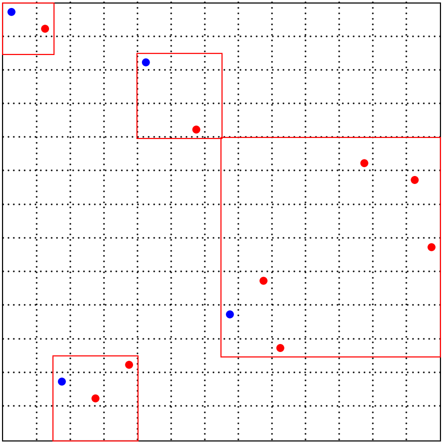
- Embed γ into β.

Generalised Approach to WFBP



- Construct new permutation ω inside β containing γ.

Generalised Approach to WFBP



- Y-profile of ω still contains γ.

The Construction Algorithm

- Begin by including the pattern of γ as we embedded it in β.
- The aim is to stop this pattern γ from dissappearing when we take the Y-profile of ω.
- Look at consecutive pairs of symbols of γ inside ω.
- Add as few symbols as possible for each of these pairs so we get a basis element of Y separating them.
- $|\omega|$ bounded $\Rightarrow \omega = \beta$ is a bounded basis element of $X \wr Y$.

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Structure from Pairs of Symbols I

Definition

The minimal block of $\sigma = s_1 \dots s_n$ containing symbols s_i and s_j (some i, j) is the smallest interval of σ containing both s_i and s_j.

- Denoted $\sigma^{\Diamond}_{i,j}$.
- $\sigma^{\Diamond}_{i,j}$ is unique for each pair (i, j).

Definition

The left extension of σ with symbols s_i, s_j is the minimal position k such that $s_i < s_k < s_j$, or $s_j < s_k < s_i$, written $L_\sigma(i, j)$.

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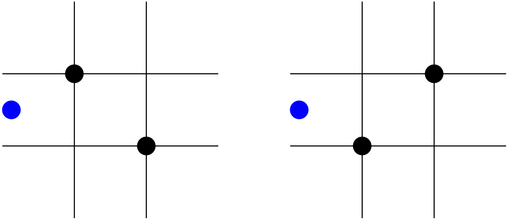

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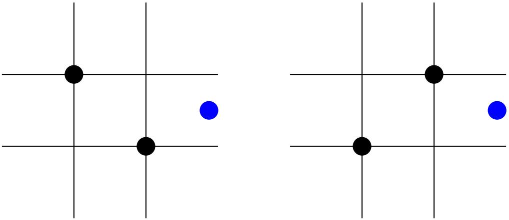

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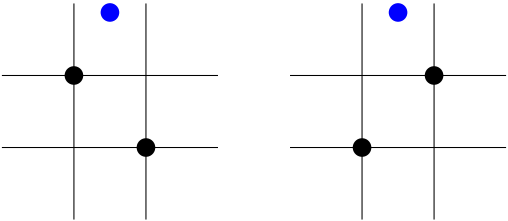

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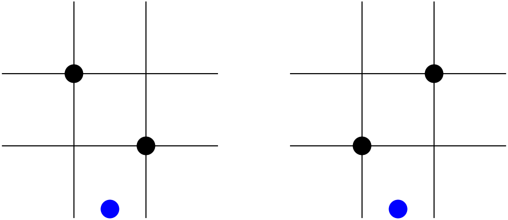

Structure from Pairs of Symbols II

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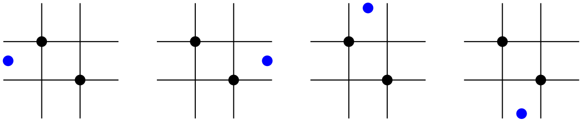

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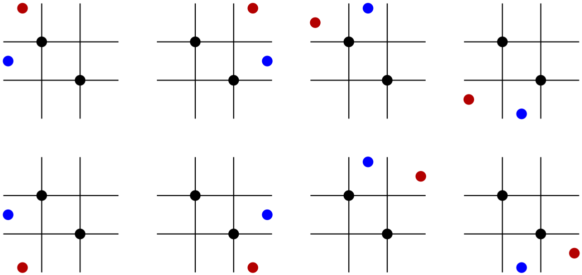

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- For a pair of positions i, j of a permutation σ, the 8 primary extensions.
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- Then the 32 tertiary extensions, . . ., the 2^{n+2} *n*-ary extensions . . .
- *n*-ary extensions may not exist. Must eventually reach the edges of the minimal block.

Structure from Pairs of Symbols III

Definition

An *n*-ary extended block of σ is the permutation formed by taking symbols with positions given by:

- An n-ary extension.
- The $(n-1)$-ary "parent" extension.

$$\vdots$$

- The primary "parent" extension, and the original i, j.

Definition

The set of n-ary extended blocks of σ on pair (i, j) is $\mathcal{E}_\sigma(i, j; n)$.
It is a subset of the generalised set of all 2^{n+2} possible n-ary extended blocks,

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WFBP & Extended Blocks

Theorem (Brignall, 2005)

Let $Y = \mathrm{Av}(B)$ be a finitely based closed class, and suppose there exists some q such that for all $\varepsilon \in \mathcal{E}(q)$, we can find a $\beta \in B$ such that $\beta \preccurlyeq \varepsilon$. Then Y possesses the WFBP.

Proof.

- The construction algorithm is bounded - can jump at most q steps before constructing a basis element of Y.
- If no q-ary extension exists then the edges of the minimal block are reached in $q - 1$ steps or fewer.
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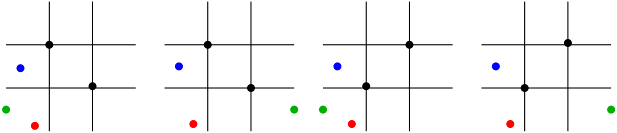
Classes with WFBP

- Stack sortable permutations, $\mathrm{Av}(231)$. Every $\varepsilon \in \mathcal{E}(3)$ involves 231.



- Separable permutations, $\mathrm{Av}(2413, 3142)$. Every $\varepsilon \in \mathcal{E}(3)$ involves 2413 or 3142.
- $\mathrm{Av}(\beta)$ for $\beta \in \{132, 312, 213\}$.
- All finite classes.
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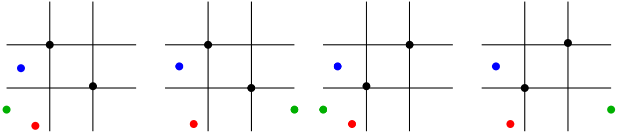

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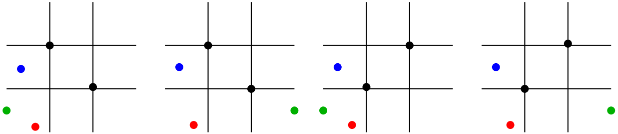

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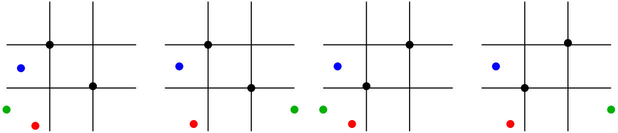

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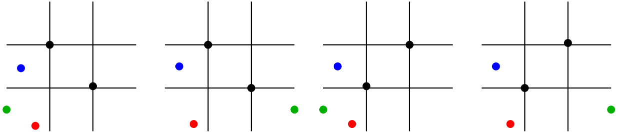

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Where Next?

- Extensions sufficient, but necessary?
- Supported by $\mathrm{Av}(123)$ which fails the theorem:

$\mathrm{Av}(42153) \wr \mathrm{Av}(123)$ is not finitely based.
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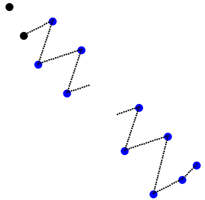

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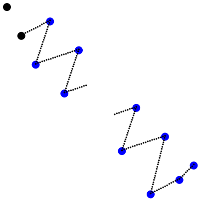

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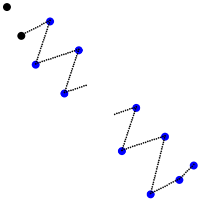

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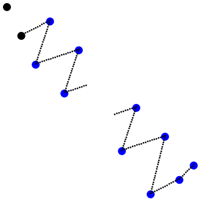

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