# Applications and Studies in Modular Decomposition

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# Outline

1. Introduction
   - Combinatorial Structures
   - Modular Decomposition
   - History
2. Applications
   - Reconstruction Conjecture
   - Permutations
3. Prime Studies
   - Fine Structure
   - Extremal Structure

# Relational Structures

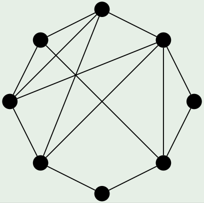
Many combinatorial objects can be described as relational structures:

- A set of points, $A$.
- A set of relations on these points.
  A $k$-ary relation $R$ – a subset of $A^k$.
- Binary relations come in many different flavours – linear, transitive, symmetric ...

Often too abstract to be useful, but (e.g.) modular decomposition is common to all of these.

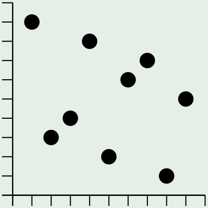
# Graphs

- Defined by a single binary symmetric relation (the edges).
- $u \sim v$ iff $v \sim u$.



# Permutations

- A permutation of length $n$ is a structure on two linear relations.



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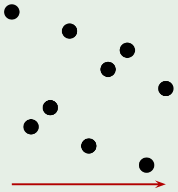

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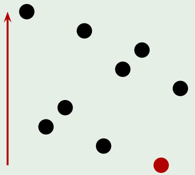

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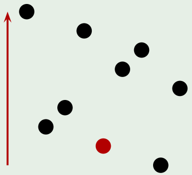

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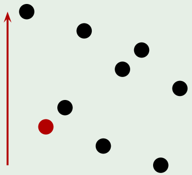

$$\pi = 9\,3\,4\,8\,2\,6\,7\,1\,5$$

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9.$
- $8 \prec 5 \prec 2$

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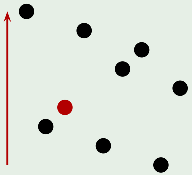

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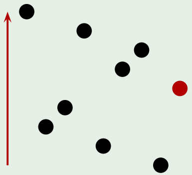

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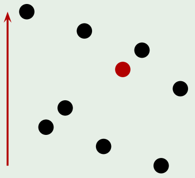

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- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6$

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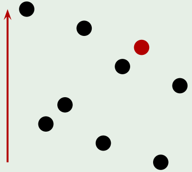

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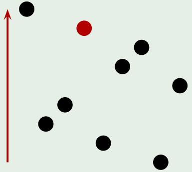

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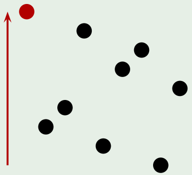

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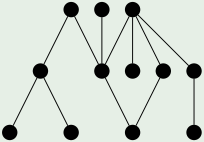
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7 \prec 4 \prec 1$

# Posets

- A binary reflexive antisymmetric transitive relation.

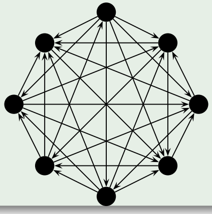


# Tournaments

- A complete oriented graph.
- As a relational structure, it is a single trichotomous binary relation:

$$x \rightarrow y, y \rightarrow x \text{ or } x = y.$$

- A competition between players: $x \rightarrow y$ means "$y$ wins."

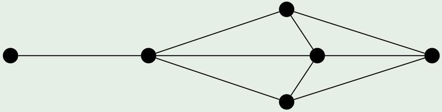


# Modules

- Module: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, intervals…
- A structure is prime if its only modules are singletons or the whole thing.
- Synonyms: Indecomposable, simple…

# Module in a graph

- $M$ is a module: neighbourhoods outside $M$ of vertices in $M$ agree: for all $u, v \in M$, $N(u) \setminus M = N(v) \setminus M$.



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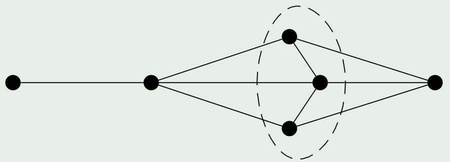

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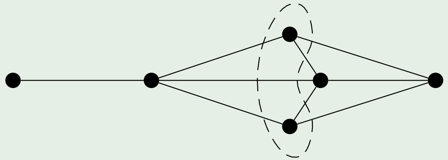

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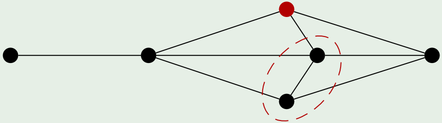
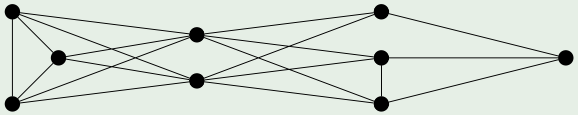


# Modular Decomposition



- Take any graph (more generally: relational structure).
- Find the maximal proper modules.

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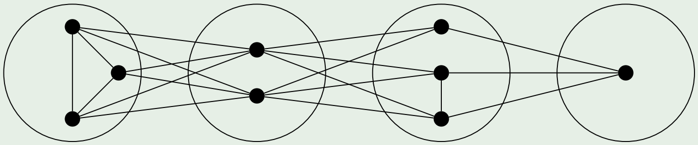

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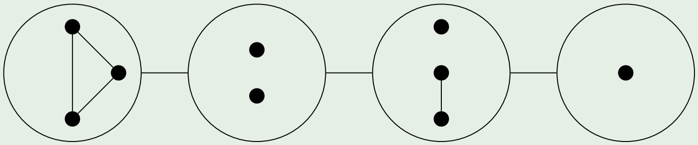

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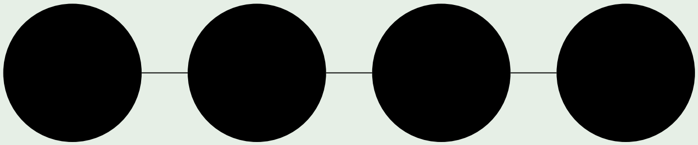
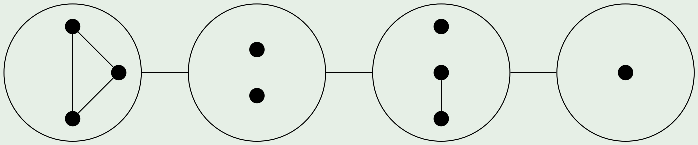


- Replace each module by a single point.
- The skeleton — $P_4$ — is prime.

# Modular Decomposition



- This is the modular decomposition (a.k.a. substitution decomposition, disjunctive decomposition, $X$-join).
- Unique unless skeleton is $K_n$ or $\overline{K_n}$.

# More formally...

**Theorem (Modular Decomposition)**

*Let $G$ be a graph. Then either*

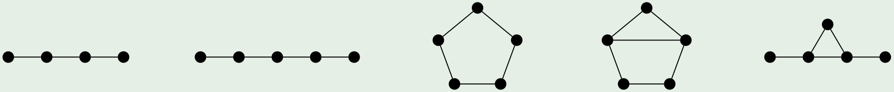
- *$G$ or $\overline{G}$ is disconnected, or*
- *$G$ has a prime skeleton, and the decomposition into maximal proper modules is unique.*

- Can be done recursively to each maximal module: modular decomposition tree.

# Prime graphs

- Modules are all singletons, or the whole graph.
- $K_2$ and $\overline{K_2}$ are special cases…
- No prime graphs on 3 vertices.

## Prime graphs on 4 and 5 vertices



# Origins

- Fraïssé (1953): gave a talk entitled “On a decomposition of relations which generalizes the sum of ordering relations”
- Gallai (1967): first article — *Transitiv orientbare Graphen*
- Feature in Lovász’s perfect graph theorem
- Möhring (1980s): game theory, combinatorial optimisation

# Graph Modular Decomposition Algorithms

- James, Stanton and Cowan, 1972: First polynomial time algorithm, $O(n^4)$.
- McConnell and Spinrad, 1994: first linear time algorithm.
- Other linear time algorithms now available.
- Parameterised complexity: recently used in kernalisation algorithms.

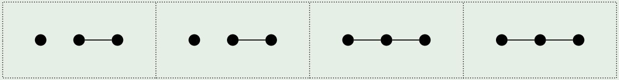
# Graph Reconstruction

The deck of a graph: $D(G) = \{* \, G - v : v \in V(G) \, *\}$.

**The Reconstruction conjecture (Ulam 1960, Kelly 1957)**

Every graph $G$ on at least 3 vertices is uniquely determined by $D(G)$.

**Example**

$D(G)$: 

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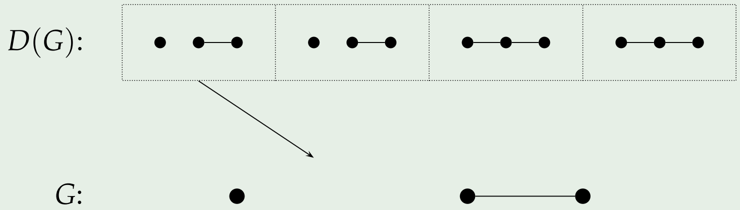

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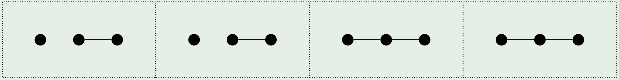

$G$:

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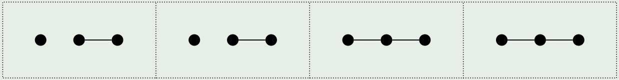

$P_4 = G$:

# Progress on RC

RC is notoriously difficult. A few highlights:

- Trees (Kelly, 1957)
- Graphs with $\geq 2$ components (folklore? See Harary 1964)
- Almost all graphs (Bollobas, 1990)
- All graphs with $\leq 11$ vertices (McKay, 1997).

Other relational structures:

- RC is True: permutations
- RC is False: digraphs, tournaments, hypergraphs, infinite graphs

# More than one component

**Proposition**

*Graphs with two or more components are reconstructible.*

**Proof.**

In $D(G)$, for each component $C$ of $G$, we have:

- $|V(G)| - |V(C)|$ copies of $C$.
- A copy of $D(C)$.

To reconstruct:

- Select a largest component in $D(G)$: must be a component of $G$.
- Remove components attributable to $C$ from $D(G)$.
- Repeat, until no more components.

□

# A special case of modular decomposition?

- $\geq 2$ components: first scenario of modular decomposition.

**Theorem (Illé, 1993)**

*$D(G)$ recognises whether $G$ is prime or not.*

Can we reconstruct decomposable (non-prime) graphs?

- Prime graphs already have a rich structure theory, so reducing RC to prime graphs could be important.

# Generalising using modular decomposition

**Lemma**

*If $G$ is decomposable, can reconstruct the skeleton.*

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*If G is decomposable, can reconstruct all the maximal proper modules.*

- So we're done, right?

. . . not quite. :(

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- How to put modules back into the skeleton?

…not quite. :(

- How to put modules back into the skeleton?

**Theorem (B., Georgiou, Waters)**

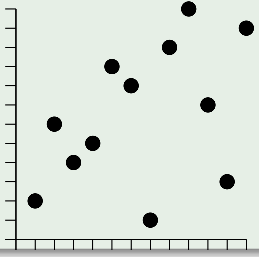
*If a decomposable graph $G$ contains a maximal module $M$ for which some $M - v$ is not a maximal module in the same orbit of the skeleton of $G$, then $G$ is reconstructible.*

- Roughly, this fails when the maximal modules of $G$ form a hereditary property.

# Intervals

- Module = interval.
- An interval of $\pi$ is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

## Example



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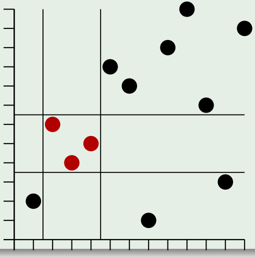

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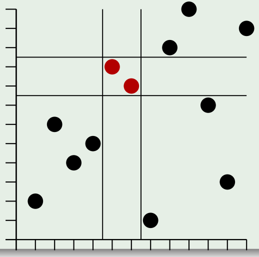


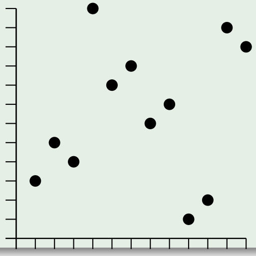
# Common Intervals and Genomics

Common interval: applies to a set $\Sigma$ of permutations.
Roughly, a set of points which each $\pi \in \Sigma$ maps to a contiguous set.

Important in gene sequence matching:

- “Reversal” = genetic mutation.
- Sorting by reversals: #steps to recover identity permutation.
- E.g. finding common ancestry of two species.

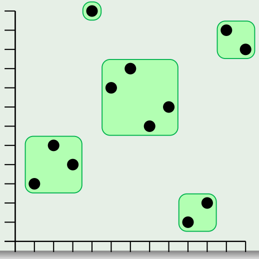
## Example



# Modular Decomposition for Permutations

- Break permutation into maximal proper intervals.

## Example



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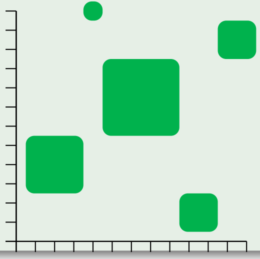

# Modular Decomposition for Permutations

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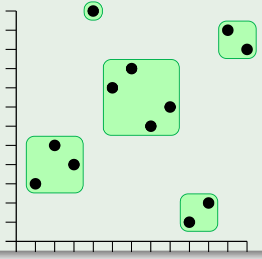
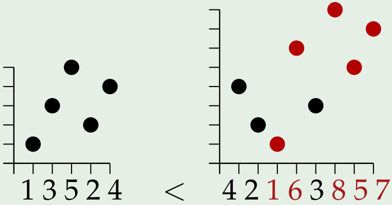
- Gives a unique prime permutation. ("simple").
- Unique unless skeleton is 12 or 21.

## Example



# Pattern avoiding permutations 101

## Example




- Pattern containment: a partial order, $\sigma \leq \pi$.

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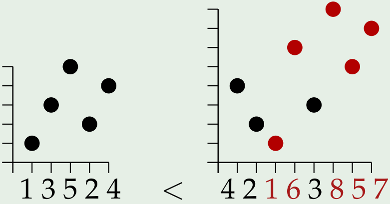


- Pattern containment: a partial order, $\sigma \leq \pi$.
- Permutation class: downset in this ordering:

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

- Avoidance: classes defined by minimal set of forbidden elements:

$$\mathcal{C} = \mathrm{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

# Uses of Modular Decomposition

Modular decomposition can help to answer questions such as:

- Enumeration: how many permutations in $\mathcal{C}$ of length $n$?
- Structure: what do permutations in $\mathcal{C}$ look like?
- Algorithms for the membership problem: is $\pi \in \mathcal{C}$?

# Finitely Many Primes

Permutation classes with only finitely many prime permutations behave well:

- Membership problem "is $\pi \in \mathcal{C}$?" answered in linear time.

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Albert and Atkinson (2005):

- They have a finite set of minimal forbidden elements.
- They are well quasi-ordered (no infinite antichains).
- They are enumerated by algebraic generating functions.

In fact...

# Algebraic Generating Functions Everywhere!

**Theorem (B., Huczynska and Vatter, 2008)**

*In a permutation class $\mathcal{C}$ with only finitely many prime permutations, the following sequences have algebraic generating functions:*

- *the number of permutations in $\mathcal{C}_n$ [Albert and Atkinson],*
- *the number of even permutations in $\mathcal{C}_n$,*
- *the number of involutions in $\mathcal{C}_n$,*
- *the number of permutations in $\mathcal{C}_n$ avoiding any finite set of blocked or barred permutations ("generalised" patterns),*
- *the number of alternating permutations in $\mathcal{C}_n$,*
- *the number of Dumont permutations in $\mathcal{C}_n$,*
- *. . . ,*
- *and any (finite) combination of the above.*

# Why study prime graphs?

Prime graphs are the elemental building blocks, simplifying studies in, e.g.

- Clique-width:
  $\mathrm{cw}(G) = \max\{\mathrm{cw}(H) : H \text{ is a prime induced subgraph of } G\}$.
- Well quasi-order: just like with permutations.
- Graph reconstruction?

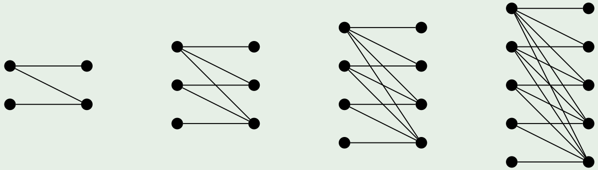
# Prime structure

**Theorem (Schmerl and Trotter, 1993)**

*Every prime graph contains a prime induced subgraph on 1 or 2 fewer vertices.*

Up to complements, one family where two vertices must be deleted:

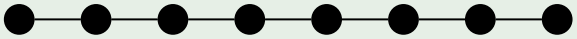
**The Half Graphs**



# Subcritical prime graphs

- Prime graph $G$: $k$-subcritical: exactly $k$ vertices for which $G - v$ is prime.
- i.e. half-graphs are "0-subcritical" (= critical).

## Paths are 2-subcritical



- Need $\geq 5$ vertices.
- Delete either leaf: get a shorter path.
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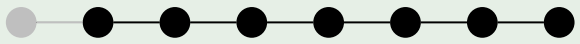
# Subcritical prime graphs

- Prime graph $G$: $k$-subcritical: exactly $k$ vertices for which $G - v$ is prime.
- i.e. half-graphs are "0-subcritical" (= critical).

## Paths are 2-subcritical



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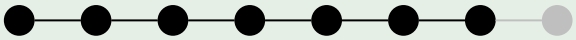

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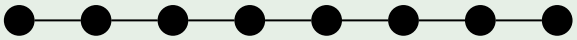

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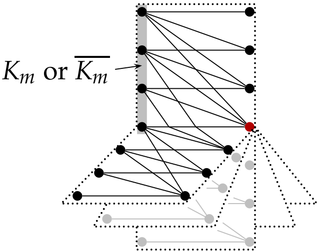
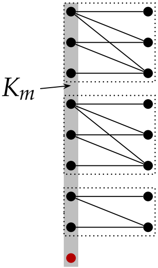
# Classifying 1-subcriticals

- Classified by Belkhechine, Boudabbous and Elayech (2010).
- B., Georgiou: shorter method, following Schmerl and Trotter.
- Structure: variations on the half graph.

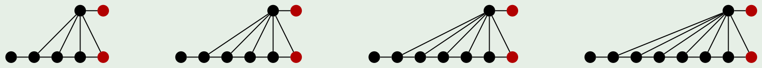


# 2-subcriticals and beyond

- Work in progress...
- Complete classification ⇒ direct proof of Illé's recognition procedure for prime graphs.
- Two basic infinite families: paths and $A_n$s:

**Family $\{A_n : n \geq 7\}$**



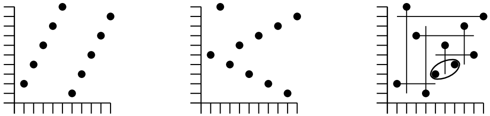
- Full range of 2-subcriticals formed from $P_n$ or $A_n$ by building "half graphs" everywhere...
- Suggests a general approach for $k$-subcriticals?

# Ramsey theory of prime graphs

Graph theoretic analogue of the following?

**Theorem (B., Huczynska and Vatter, 2008)**

*Every prime permutation of length at least* $2(256k^8)^{2k}$ *contains a prime permutation of length at least* $2k$ *from one of three families.*



# Why?

For permutations, we have a decision procedure:

**Theorem (B., Ruškuc and Vatter, 2008)**

*It is decidable if a permutation class defined by a finite set of forbidden elements contains only finitely many prime permutations.*

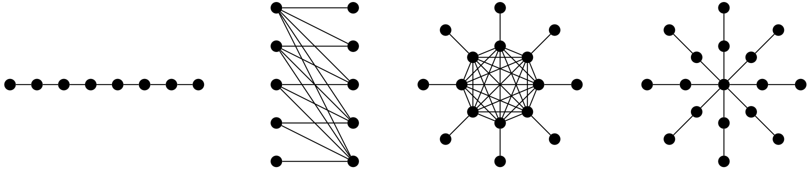
**Theorem (Bassino, Bouvel, Pierrot, Rossin, 2011+)**

*Decision procedure can be done in polynomial time (w.r.t. forbidden elements).*

Similar results would follow for hereditary properties of graphs.

# Probable unavoidable substructures

The list of prime structures probably includes:



- Permutation case does not seem to translate.
- Can $k$-subcriticals help?

Thanks!