Infinite Antichains: from Permutations to Graphs

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Orderings on Structures

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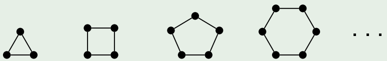
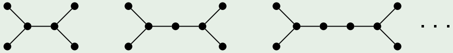
- Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, ...
- Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment, ...
- Does your ordering contain infinite antichains?
  i.e. an infinite set of pairwise incomparable elements.

Example (Induced subgraph antichains)

Cycles:



"Split end" graphs:



When are there antichains?

No infinite antichains – well-quasi-ordered.

- Words over a finite alphabet with subword ordering [Higman, 1952].
- Trees ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- Graphs closed under minors [Robertson and Seymour, 1983—2004].

Infinite antichains.

- Graphs closed under induced subgraphs (or merely subgraphs).
- Permutations closed under containment.
- Tournaments, digraphs, posets, . . . with their natural induced substructure ordering.

Downsets

Question

In your favourite ordering, which downsets contain infinite antichains?

- Downset (or hereditary property): set $\mathcal{P}$ of objects such that

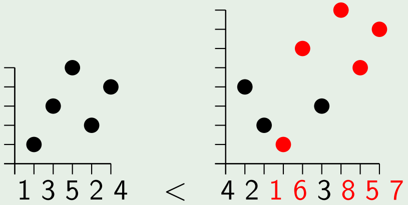
$$G \in \mathcal{P} \text{ and } H \leq G \text{ implies } H \in \mathcal{P}.$$

  e.g. triangle-free graphs — (induced) subgraph ordering.
- For permutation containment, these are called permutation classes.
  e.g. the class of "stack sortable" permutations.

Permutation Containment

- Write permutations in one-line notation, e.g. $\tau = 13524$.
- A permutation $\tau = \tau(1)\cdots\tau(k)$ is contained in the permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ if there exists a subsequence $\sigma(i_1)\sigma(i_2)\cdots\sigma(i_k)$ order isomorphic to τ.

Example



Permutation Classes

- Containment is a partial order on the set of all permutations.
- Recall: downsets are permutation classes. i.e. $\pi \in \mathcal{C}$ and $\sigma \leq \pi$ implies $\sigma \in \mathcal{C}$.
- Each class has a unique set of minimal forbidden elements. Write

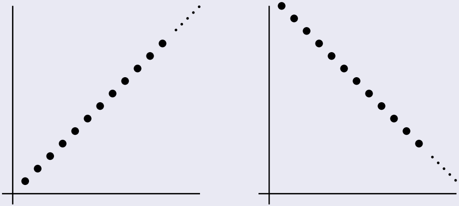
$$\mathcal{C} = \mathsf{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

B is (unfortunately) called the basis.

Easy Examples

- $\text{Av}(21) = \{1, 12, 123, 1234, \ldots\}$, the increasing permutations.
- $\text{Av}(12) = \{1, 21, 321, 4321, \ldots\}$, the decreasing permutations.

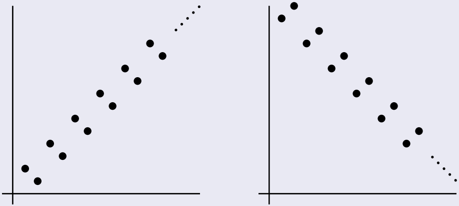
Typical Elements



Easy Examples

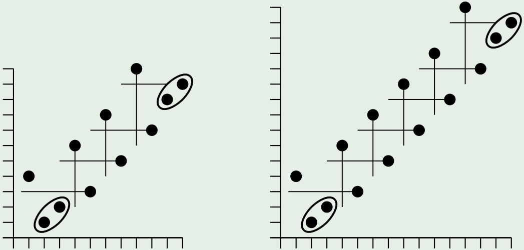
- $\oplus 21 = \text{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots\}$.
- $\ominus 12 = \text{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots\}$.

Typical Elements



Increasing Oscillations: a Permutation Antichain

Two typical elements



- Need to show there is no embedding of one in the other.

Increasing Oscillations: a Permutation Antichain

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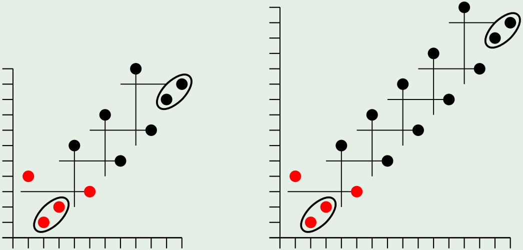

- Anchor: bottom copies of 4123 must match up.

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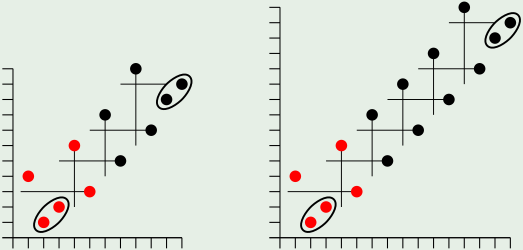

- Each point is matched in turn.

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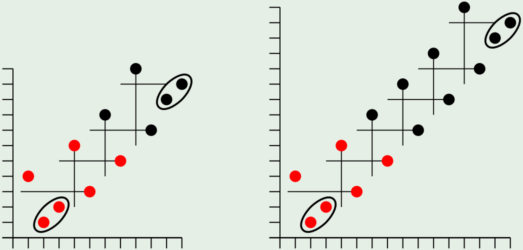

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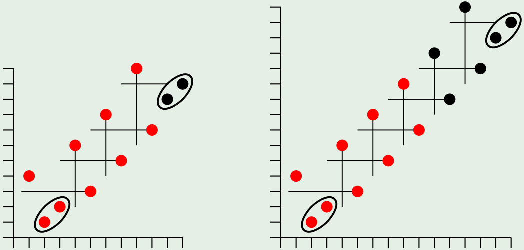

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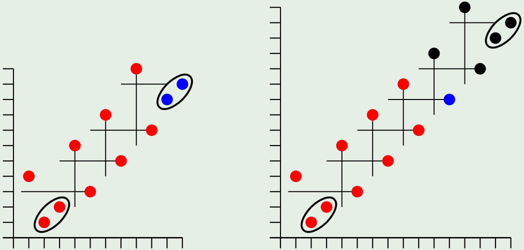

- Last pair cannot be embedded.

Increasing Oscillations: a Permutation Antichain

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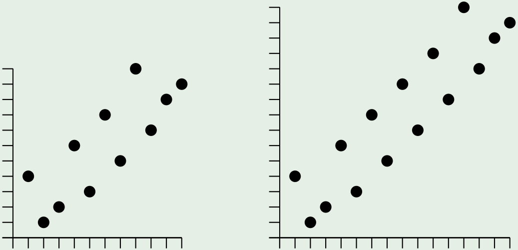

- Alternatively, make a graph: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$

Increasing Oscillations: a Permutation Antichain

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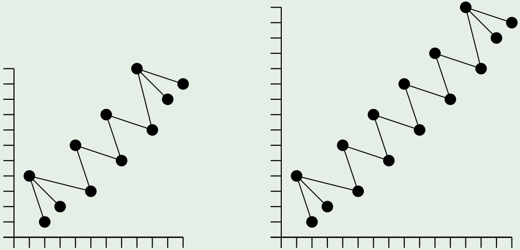
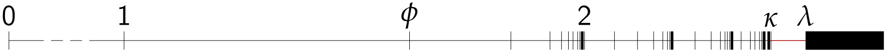


- The split end antichain!

Aside: Asymptotic Enumeration

- $\mathcal{C}_n$ – permutations in $\mathcal{C}$ of length n.
- Growth rate of $\mathcal{C}$ is $\lim_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}$ (if it exists).
- Below $\kappa \approx 2.20557$, growth rates exist and can be characterised [Vatter, 2007+]:



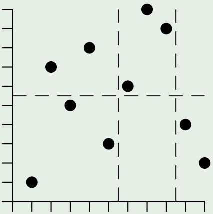
- At κ, we find the increasing oscillating antichain, and hence uncountably many permutation classes.
  The proof uses grid classes (more later).
- Above $\lambda \approx 2.48188$, every real number is the growth rate of a permutation class [Vatter, 2010].
  The proof builds classes based on this antichain.
- From order to chaos: What lies between κ and λ?

Grid Classes

- Hot topic: Crucial tool to study the structure of classes.
- Matrix $\mathcal{M}$ whose entries are (infinite) permutation classes.
- Grid($\mathcal{M}$) the grid class of $\mathcal{M}$: all permutations which can be "gridded" so each cell satisfies constraints of $\mathcal{M}$.

Example

- Let $\mathcal{M} = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(231) & \emptyset \\ \mathrm{Av}(123) & \emptyset & \mathrm{Av}(12) \end{pmatrix}$.



$\in$ Grid($\mathcal{M}$)

Grid classes of graphs?

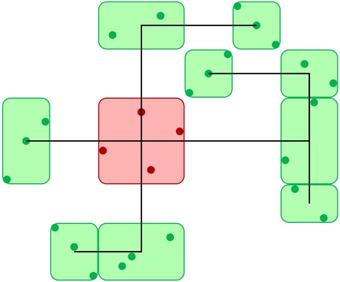
There are some related concepts in graph theory:

- Split graphs: graphs that can be partitioned into a clique and an independent set.
- Canonical properties, used in asymptotic enumeration ("speeds") of hereditary properties [Balogh, Bollobás and Weinreich]
- Matrix partitions of graphs [Feder and Hell]

Grid Classes and Well-quasi-order

[B., 2009+]

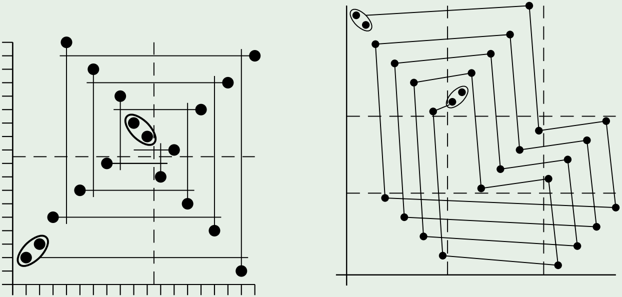
- A general construction for infinite antichains in all but one family of grid classes.
- Within this family, proof that certain grid classes are well-quasi-ordered.



Antichains round Cycles

- Murphy and Vatter, 2003: Build an antichain by placing points sequentially around a "cycle".

Two examples



- N.B. Each non-empty cell is monotone.

There and Back Again Antichains

- B, 2009+: Build an antichain on a path, providing you can “turn around” at each end.

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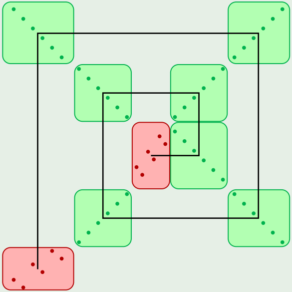

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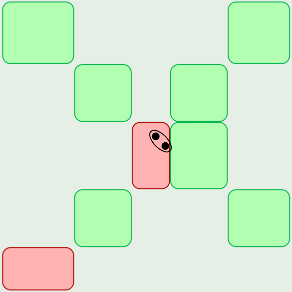

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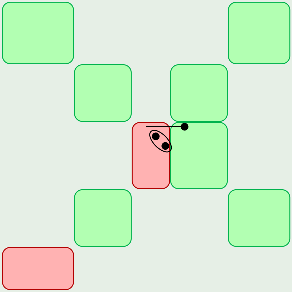

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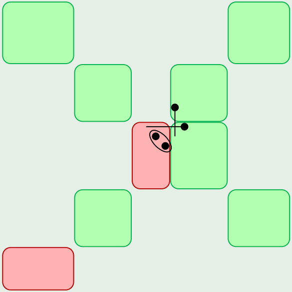

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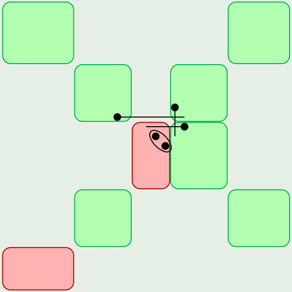

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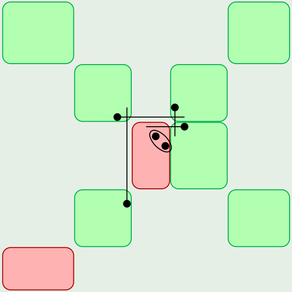

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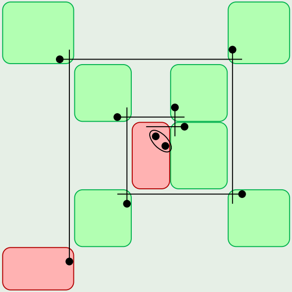

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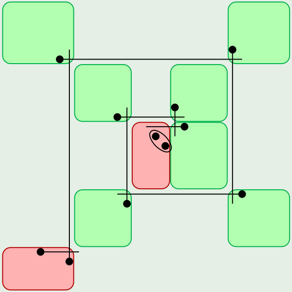

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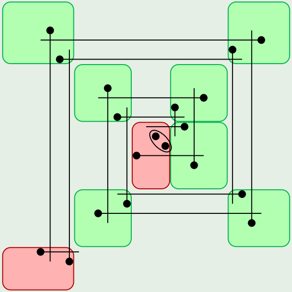

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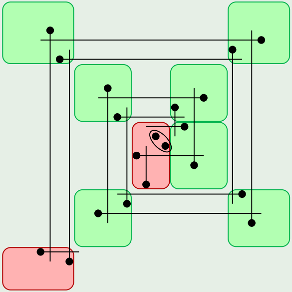

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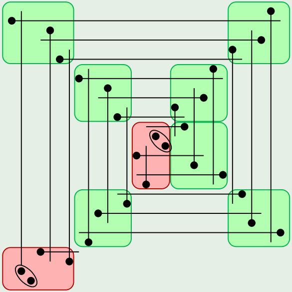


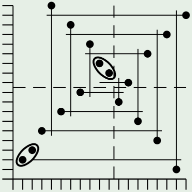
To Graphs. . .

- Two cheap results...

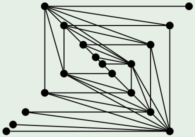
Conjecture (Ding, 1992)

The hereditary property of permutation graphs that do not contain (as an induced subgraph) a path or the complement of a path on $n \geq 5$ vertices is well-quasi-ordered.

Counterexample



becomes (roughly)



Double-split graphs

- Double-split graph: partitions into a matching, and the complement of a matching.
- As seen in the strong perfect graph theorem [Chudnovsky, Robertson, Seymour and Thomas, 2006].
- Hereditary property: take the downward closure. It is characterised by 44 minimal forbidden graphs [Alexeev, Fradkin, Kim, 2010]

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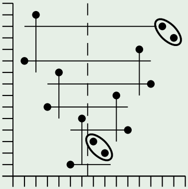
- ... but it is not well-quasi-ordered:

Turn this into a graph



Thanks!