Infinite Antichains: from Permutations to Graphs

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Pick your favourite family of combinatorial structures. E.g. graphs, permutations, tournaments, posets, …
Orderings on Structures

- Pick your favourite family of combinatorial structures.
  E.g. graphs, permutations, tournaments, posets, . . .
- Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment, . . .
Orderings on Structures

- Pick your favourite family of combinatorial structures.
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- Give your family an ordering.
  E.g. graph minor, induced subgraph, permutation containment, ...

- Does your ordering contain infinite antichains?
  i.e. an infinite set of pairwise incomparable elements.

Example (Induced subgraph antichains)

Cycles:

```
triangle
square
circle
circle
```

“Split end” graphs:

```
triangle
square
square
```

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When are there antichains?

No infinite antichains – well-quasi-ordered.
- **Words** over a finite alphabet with subword ordering [Higman, 1952].
- **Trees** ordered by topological minors [Kruskal 1960; Nash-Williams, 1963]
- Graphs closed under **minors** [Robertson and Seymour, 1983—2004].

Infinite antichains.
- Graphs closed under **induced subgraphs** (or merely subgraphs).
- Permutations closed under **containment**.
- Tournaments, digraphs, posets, ... with their natural **induced substructure** ordering.
Downsets

Question

*In your favourite ordering, which downsets contain infinite antichains?*

- Downset (or **hereditary property**): set $\mathcal{P}$ of objects such that

  \[ G \in \mathcal{P} \text{ and } H \leq G \text{ implies } H \in \mathcal{P}. \]

  e.g. triangle-free graphs — (induced) subgraph ordering.

- For permutation containment, these are called **permutation classes**.
  e.g. the class of “stack sortable” permutations.
Permutation Containment

- Write permutations in one-line notation, e.g. \( \tau = 13524 \).
- A permutation \( \tau = \tau(1) \cdots \tau(k) \) is contained in the permutation \( \sigma = \sigma(1)\sigma(2) \cdots \sigma(n) \) if there exists a subsequence \( \sigma(i_1)\sigma(i_2) \cdots \sigma(i_k) \) order isomorphic to \( \tau \).

Example

\[ \begin{array}{cccccc}
1 & 3 & 5 & 2 & 4 \\
\end{array} \quad < \quad \begin{array}{cccccc}
4 & 2 & 1 & 6 & 3 & 8 & 5 & 7 \\
\end{array} \]
Containment is a **partial order** on the set of all permutations.

Recall: downsets are permutation classes. i.e. \( \pi \in \mathcal{C} \) and \( \sigma \leq \pi \) implies \( \sigma \in \mathcal{C} \).

Each class has a **unique** set of minimal forbidden elements. Write

\[
\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.
\]

\( B \) is (unfortunately) called the **basis**.
Easy Examples

- \( \text{Av}(21) = \{1, 12, 123, 1234, \ldots\} \), the increasing permutations.
- \( \text{Av}(12) = \{1, 21, 321, 4321, \ldots\} \), the decreasing permutations.

Typical Elements
Easy Examples

\( \ominus 21 = \text{Av}(321, 312, 231) = \{1, 12, 21, 123, 132, 213, \ldots \} \).

\( \ominus 12 = \text{Av}(123, 213, 132) = \{1, 12, 21, 231, 312, 321, \ldots \} \).
Increasing Oscillations: a Permutation Antichain

Two typical elements

Need to show there is no embedding of one in the other.
Two typical elements

- Anchor: bottom copies of 4123 must match up.
Increasing Oscillations: a Permutation Antichain

Two typical elements

- Each point is matched in turn.
Increasing Oscillations: a Permutation Antichain

Two typical elements

- Each point is matched in turn.
Each point is matched in turn.
Increasing Oscillations: a Permutation Antichain

Two typical elements

Last pair cannot be embedded.
Increasing Oscillations: a Permutation Antichain

Two typical elements

Alternatively, make a graph: for $i < j$, $i \sim j$ iff $\pi(i) > \pi(j)$
Increasing Oscillations: a Permutation Antichain

Two typical elements

- The split end antichain!
Aside: Asymptotic Enumeration

- $C_n$ – permutations in $C$ of length $n$.
- **Growth rate** of $C$ is $\lim_{n \to \infty} \sqrt[n]{|C_n|}$ (if it exists).
- Below $\kappa \approx 2.20557$, growth rates exist and can be characterised [Vatter, 2007+]:

\[
\begin{array}{cccccc}
0 & 1 & \phi & 2 & \kappa & \lambda \\
\hline
\end{array}
\]

- At $\kappa$, we find the increasing oscillating antichain, and hence uncountably many permutation classes. The proof uses grid classes (more later).
- Above $\lambda \approx 2.48188$, every real number is the growth rate of a permutation class [Vatter, 2010]. The proof builds classes based on this antichain.
- From order to chaos: What lies between $\kappa$ and $\lambda$?
Grid Classes

- Hot topic: Crucial tool to study the structure of classes.
- **Matrix** $\mathcal{M}$ whose entries are (infinite) permutation classes.
- $\text{Grid}(\mathcal{M})$ the **grid class** of $\mathcal{M}$: all permutations which can be “gridded” so each cell satisfies constraints of $\mathcal{M}$.

**Example**

- Let $\mathcal{M} = \begin{pmatrix} \text{Av}(21) & \text{Av}(231) & \emptyset \\ \text{Av}(123) & \emptyset & \text{Av}(12) \end{pmatrix}$.
There are some related concepts in graph theory:

- **Split graphs**: graphs that can be partitioned into a clique and an independent set.
- **Canonical properties**, used in asymptotic enumeration ("speeds") of hereditary properties [Balogh, Bollobás and Weinreich]
- **Matrix partitions** of graphs [Feder and Hell]
[B., 2009+]

- A general construction for infinite antichains in all but one family of grid classes.
- Within this family, proof that certain grid classes are well-quasi-ordered.
Antichains round Cycles

- Murphy and Vatter, 2003: Build an antichain by placing points sequentially around a “cycle”.

Two examples

- N.B. Each non-empty cell is monotone.
B, 2009+: Build an antichain on a path, providing you can “turn around” at each end.
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There and Back Again Antichains

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Example

```

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Example
To Graphs...

- Two cheap results...

**Conjecture (Ding, 1992)**

The hereditary property of permutation graphs that do not contain (as an induced subgraph) a path or the complement of a path on \( n \geq 5 \) vertices is well-quasi-ordered.

**Counterexample**

becomes (roughly)
Double-split graphs

- **Double-split graph**: partitions into a matching, and the complement of a matching.
- As seen in the strong perfect graph theorem [Chudnovsky, Robertson, Seymour and Thomas, 2006].
- Hereditary property: take the **downward closure**. It is characterised by 44 minimal forbidden graphs [Alexeev, Fradkin, Kim, 2010]
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- Hereditary property: take the **downward closure**. It is characterised by 44 minimal forbidden graphs [Alexeev, Fradkin, Kim, 2010].
- … but it is not well-quasi-ordered:

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**Turn this into a graph**

![Graph](image-url)
Thanks!