

Applications and Studies in Modular Decomposition

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Outline

- 1 Introduction
 - Combinatorial Structures
 - Modular Decomposition
 - History
- 2 Applications
 - Reconstruction Conjecture
 - Permutations
- 3 Prime Studies
 - Fine Structure
 - Extremal Structure

Relational Structures

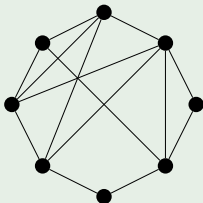
Many combinatorial objects can be described as **relational structures**:

- A **set of points**, A .
- A **set of relations** on these points.
A k -ary relation R – a subset of A^k .
- **Binary** relations come in many different flavours – linear, transitive, symmetric ...

Often too abstract to be useful, but (e.g.) **modular decomposition** is common to all of these.

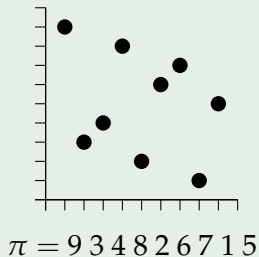
Graphs

- Defined by a single binary symmetric relation (the edges).
- $u \sim v$ iff $v \sim u$.



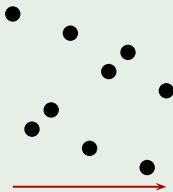
Permutations

- A permutation of length n is a structure on **two linear relations**.



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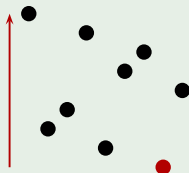


$$\pi = 934826715$$

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$.

Permutations

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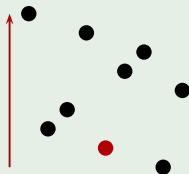


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- 8

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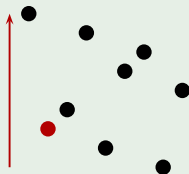


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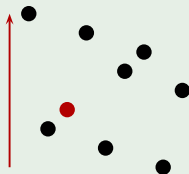


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- $8 \prec 5 \prec 2$

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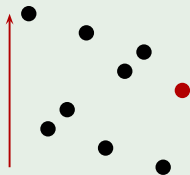


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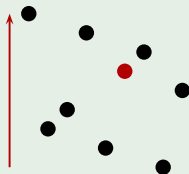


$$\pi = 9\ 3\ 4\ 8\ 2\ 6\ 7\ 1\ 5$$

- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$.
- $8 \prec 5 \prec 2 \prec 3 \prec 9$

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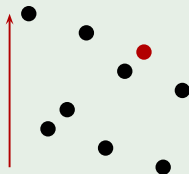


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- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$.
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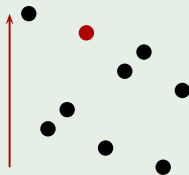


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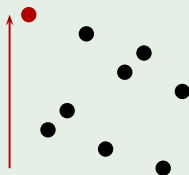


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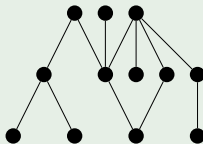
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- $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$.
- $8 \prec 5 \prec 2 \prec 3 \prec 9 \prec 6 \prec 7 \prec 4 \prec 1$

- A binary reflexive antisymmetric transitive relation.

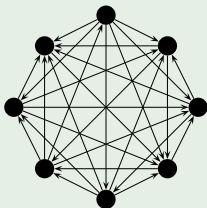


Tournaments

- A complete oriented graph.
- As a relational structure, it is a single trichotomous binary relation:

$$x \rightarrow y, y \rightarrow x \text{ or } x = y.$$

- A **competition** between players: $x \rightarrow y$ means “ y wins.”

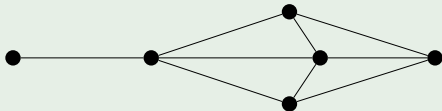


Modules

- **Module**: set of points which “look” at every other point in the same way.
- Synonyms: Autonomous sets, blocks, bound sets, closed sets, clumps, convex sets, intervals...
- A structure is **prime** if its only modules are singletons or the whole thing.
- Synonyms: Indecomposable, simple...

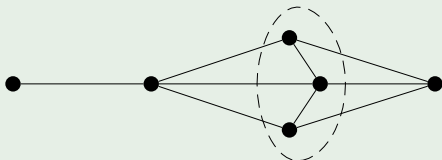
Module in a graph

- M is a **module**: neighbourhoods outside M of vertices in M agree:
for all $u, v \in M$, $N(u) \setminus M = N(v) \setminus M$.



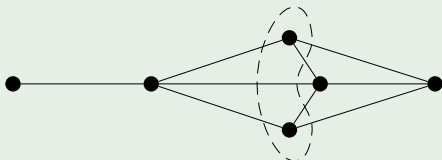
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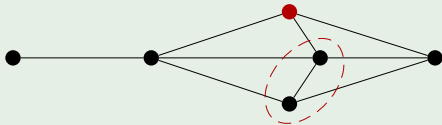
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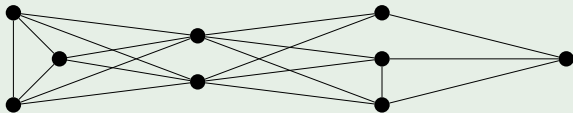


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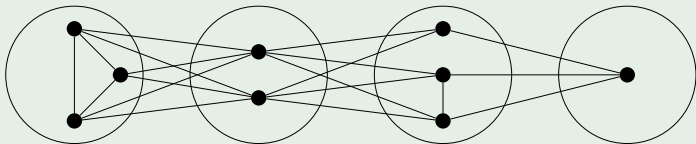


Modular Decomposition



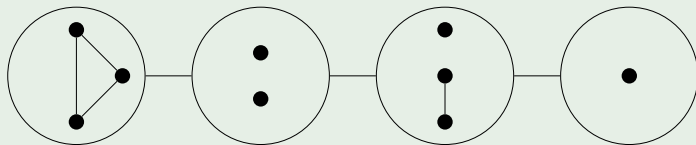
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- Find the **maximal proper modules**.

Modular Decomposition



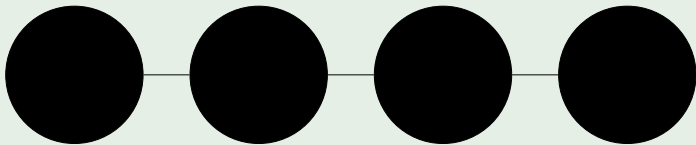
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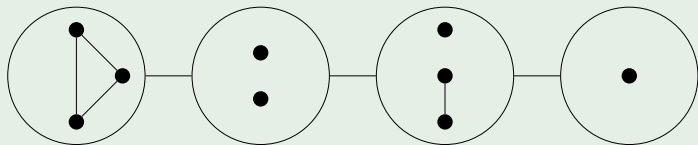
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Modular Decomposition



- Replace each module by a **single point**.
- The **skeleton** — P_4 — is **prime**.

Modular Decomposition



- This is the **modular decomposition** (a.k.a. substitution decomposition, disjunctive decomposition, X-join).
- **Unique** unless skeleton is K_n or \overline{K}_n .

More formally...

Theorem (Modular Decomposition)

Let G be a graph. Then either

- G or \overline{G} is *disconnected*, or
 - G has a prime skeleton, and the decomposition into maximal proper modules is *unique*.
-
- Can be done recursively to each maximal module: **modular decomposition tree**.

Prime graphs

- Modules are all singletons, or the whole graph.
- K_2 and $\overline{K_2}$ are special cases...
- No prime graphs on 3 vertices.

Prime graphs on 4 and 5 vertices



Origins

- Fraïssé (1953): gave a talk entitled “On a decomposition of relations which generalizes the sum of ordering relations”
- Gallai (1967): first article — *Transitiv orientbare Graphen*
- Feature in Lovász’s **perfect graph theorem**
- Möhring (1980s): game theory, combinatorial optimisation

Graph Modular Decomposition Algorithms

- James, Stanton and Cowan, 1972: First polynomial time algorithm, $O(n^4)$.
- McConnell and Spinrad, 1994: first **linear time** algorithm.
- Other linear time algorithms now available.
- **Parameterised complexity**: recently used in kernalisation algorithms.

Graph Reconstruction

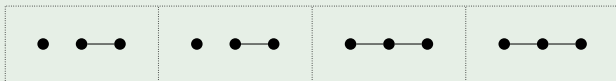
The **deck** of a graph: $D(G) = \{ * G - v : v \in V(G) * \}$.

The Reconstruction conjecture (Ulam 1960, Kelly 1957)

Every graph G on at least 3 vertices is uniquely determined by $D(G)$.

Example

$D(G)$:



Graph Reconstruction

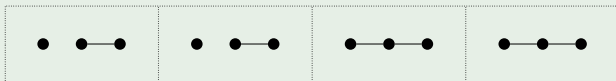
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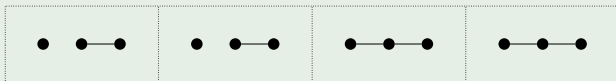
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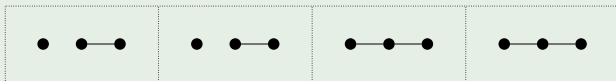
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Example

$D(G)$:



$P_4 = G$:



Progress on RC

RC is notoriously difficult. A few highlights:

- Trees (Kelly, 1957)
- Graphs with ≥ 2 components (folklore? See Harary 1964)
- Almost all graphs (Bollobas, 1990)
- All graphs with ≤ 11 vertices (McKay, 1997).

Other relational structures:

- RC is **True**: permutations
- RC is **False**: digraphs, tournaments, hypergraphs, infinite graphs

More than one component

Proposition

Graphs with two or more components are reconstructible.

Proof.

In $D(G)$, for each component C of G , we have:

- $|V(G)| - |V(C)|$ copies of C .
- A copy of $D(C)$.

To reconstruct:

- Select a largest component in $D(G)$: must be a component of G .
- Remove components **attributable** to C from $D(G)$.
- Repeat, until no more components.



A special case of modular decomposition?

- ≥ 2 components: first scenario of modular decomposition.

Theorem (Illé, 1993)

D(G) recognises whether G is prime or not.

Can we reconstruct decomposable (non-prime) graphs?

- **Prime graphs** already have a rich structure theory, so reducing RC to prime graphs could be important.

Generalising using modular decomposition

Lemma

If G is decomposable, can reconstruct the *skeleton*.

Generalising using modular decomposition

Lemma

If G is decomposable, can reconstruct the *skeleton*.

Lemma

If G is decomposable, can reconstruct all the *maximal proper modules*.

- So we're done, right?

...not quite. :(

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- How to put modules back into the skeleton?

...not quite. :(

- How to put modules back into the skeleton?

Theorem (B., Georgiou, Waters)

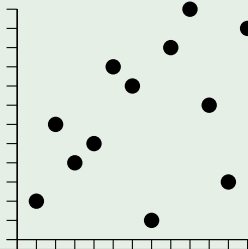
If a decomposable graph G contains a maximal module M for which some $M - v$ is not a maximal module in the same orbit of the skeleton of G , then G is reconstructible.

- Roughly, this **fails** when the maximal modules of G form a **hereditary property**.

Intervals

- Module = interval.
- An **interval** of π is a set of contiguous indices $I = [a, b]$ such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

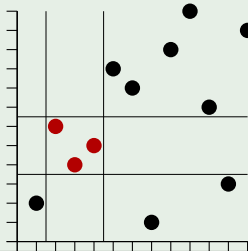
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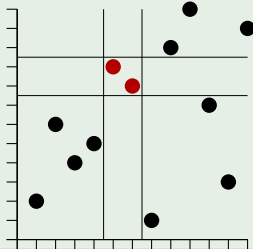
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Example



Common Intervals and Genomics

Common interval: applies to a set Σ of permutations.

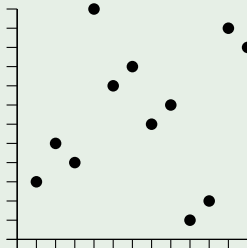
Roughly, a set of points which each $\pi \in \Sigma$ maps to a contiguous set.

Important in **gene sequence matching:**

- “Reversal” = genetic mutation.
- **Sorting by reversals:** #steps to recover identity permutation.
- E.g. finding common ancestry of two species.

Modular Decomposition for Permutations

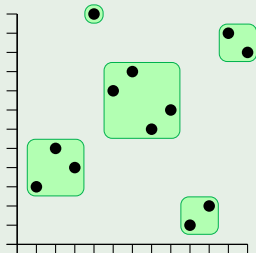
Example



Modular Decomposition for Permutations

- Break permutation into **maximal proper intervals**.

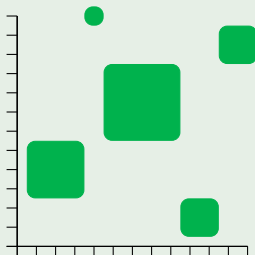
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Modular Decomposition for Permutations

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- Gives a **unique** prime permutation. (“simple”).

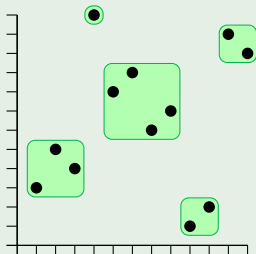
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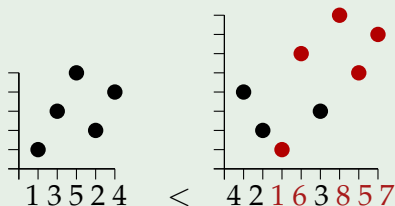
- Break permutation into **maximal proper intervals**.
- Gives a **unique** prime permutation. (“simple”).
- **Unique** unless skeleton is 12 or 21.

Example



Pattern avoiding permutations 101

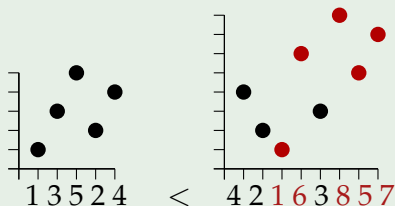
Example



- **Pattern containment:** a partial order, $\sigma \leq \pi$.

Pattern avoiding permutations 101

Example



- **Pattern containment:** a partial order, $\sigma \leq \pi$.
- **Permutation class:** downset in this ordering:

$$\pi \in \mathcal{C} \text{ and } \sigma \leq \pi \text{ implies } \sigma \in \mathcal{C}.$$

- **Avoidance:** classes defined by minimal set of forbidden elements:

$$\mathcal{C} = \text{Av}(B) = \{\pi : \beta \not\leq \pi \text{ for all } \beta \in B\}.$$

Uses of Modular Decomposition

Modular decomposition can help to answer questions such as:

- **Enumeration**: how many permutations in \mathcal{C} of length n ?
- **Structure**: what do permutations in \mathcal{C} look like?
- Algorithms for the **membership problem**: is $\pi \in \mathcal{C}$?

Finitely Many Primes

Permutation classes with only **finitely many prime permutations** behave well:

- **Membership problem** “is $\pi \in \mathcal{C}$?” answered in linear time.

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Albert and Atkinson (2005):

- They have a **finite set of minimal forbidden elements**.
- They are **well quasi-ordered** (no infinite antichains).
- They are enumerated by **algebraic generating functions**.

In fact...

Algebraic Generating Functions Everywhere!

Theorem (B., Huczynska and Vatter, 2008)

In a permutation class \mathcal{C} with only finitely many prime permutations, the following sequences have algebraic generating functions:

- the number of *permutations* in \mathcal{C}_n [Albert and Atkinson],
- the number of *even* permutations in \mathcal{C}_n ,
- the number of *involutions* in \mathcal{C}_n ,
- the number of permutations in \mathcal{C}_n avoiding any finite set of *blocked* or *barred* permutations (“generalised” patterns),
- the number of *alternating* permutations in \mathcal{C}_n ,
- the number of *Dumont* permutations in \mathcal{C}_n ,
- ... ,
- and any (finite) *combination* of the above.

Why study prime graphs?

Prime graphs are the elemental **building blocks**, simplifying studies in, e.g.

- Clique-width:
$$\text{cw}(G) = \max\{\text{cw}(H) : H \text{ is a prime induced subgraph of } G\}.$$
- Well quasi-order: just like with permutations.
- Graph reconstruction?

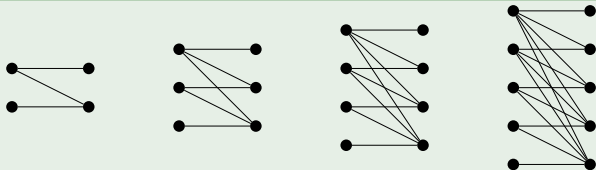
Prime structure

Theorem (Schmerl and Trotter, 1993)

Every prime graph contains a prime induced subgraph on 1 or 2 fewer vertices.

Up to complements, one family where two vertices must be deleted:

The Half Graphs



Subcritical prime graphs

- Prime graph G : **k -subcritical**: exactly k vertices for which $G - v$ is prime.
- i.e. half-graphs are “0-subcritical” (= critical).

Paths are 2-subcritical

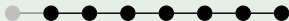


- Need ≥ 5 vertices.
- Delete **either leaf**: get a shorter path.
- Delete any other vertex: graph is **disconnected**.

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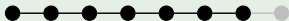


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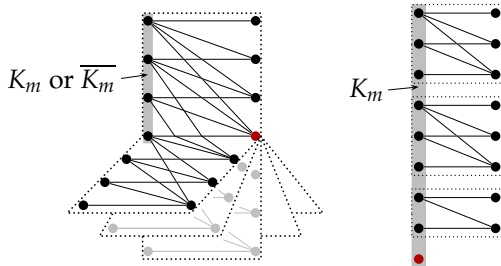
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Classifying 1-subcriticals

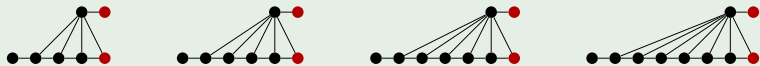
- **Classified** by Belkhechine, Boudabbous and Elayech (2010).
- B., Georgiou: shorter method, following Schmerl and Trotter.
- Structure: variations on the half graph.



2-subcriticals and beyond

- Work in progress...
- Complete classification \Rightarrow direct proof of Illé's **recognition procedure** for prime graphs.
- Two basic infinite families: **paths** and A_n s:

Family $\{A_n : n \geq 7\}$



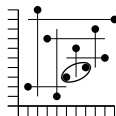
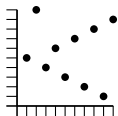
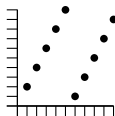
- Full range of 2-subcriticals formed from P_n or A_n by building “half graphs” everywhere...
- Suggests a general approach for **k -subcriticals**?

Ramsey theory of prime graphs

Graph theoretic analogue of the following?

Theorem (B., Huczynska and Vatter, 2008)

Every prime permutation of length at least $2(256k^8)^{2k}$ contains a prime permutation of length at least $2k$ from one of three families.



Why?

For permutations, we have a **decision procedure**:

Theorem (B., Ruškuc and Vatter, 2008)

It is decidable if a permutation class defined by a finite set of forbidden elements contains only finitely many prime permutations.

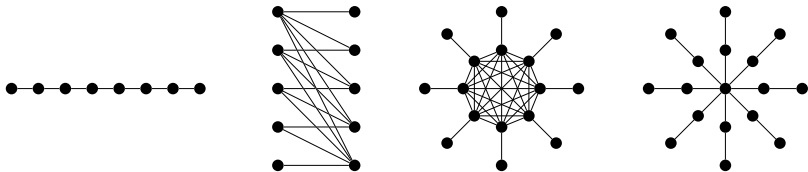
Theorem (Bassino, Bouvel, Pierrot, Rossin, 2012+)

*Decision procedure can be done in **polynomial time** (w.r.t. forbidden elements).*

Similar results would follow for **hereditary properties** of graphs.

Probable unavoidable substructures

The list of prime structures should include:



- Permutation case does not seem to translate.
- Can k -subcriticals help?

Thanks!